

GENERALIZED VECTORIAL LEBESGUE AND BOCHNER INTEGRATION THEORY

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ABSTRACT. This paper contains a development of the Theory of Lebesgue and Bochner spaces of summable functions. It represents a synthesis of the results due to H. Lebesgue, S. Banach, S. Bochner, G. Fubini, S. Saks, F. Riesz, N. Dunford, P. Halmos, and other contributors to this theory.

The construction of the theory is based on the notion of a measure on a prering of sets in any abstract space X . No topological structure of the space X is required for the development of the theory.

Measures on prerings generalize the notion of abstract Lebesgue measures. These measures are readily available and it is not necessary to extend them beforehand onto a sigma-ring for the development of the theory.

The basic tool in the development of the theory is the construction and characterization of Lebesgue-Bochner spaces of summable functions as in the paper of Bogdanowicz, "A Generalization of the Lebesgue-Bochner-Stieltjes Integral and a New Approach to the Theory of Integration", Proc. of Nat. Acad. Sci. USA, Vol. 53, No. 3, (1965), p. 492–498

1. INTRODUCTION

In this paper we shall present a development of the theory of Lebesgue and Bochner spaces of summable functions and prove the fundamental theorems of the theory.

The development of the integration theory beyond the classical Riemann integral is important for advancements in modern theory of differential equations, theory of generalized functions, theory of operators, probability, optimal control, and most of all in theoretical physics.

Generalized functions introduced into mathematics by P. Dirac and put on precise mathematical footing by L. Schwartz [22], I. Gelfand and G. Shilov [17], turned out to be essential in analysis of flows of matter endowed with mass. For reference see Bogdan [10].

This paper represents a partial synthesis of the results due to H. Lebesgue [19], S. Banach [1], S. Bochner [3], S. Saks [25], F. Riesz [24], N. Dunford [12], and P. Halmos [18] and other contributors to this theory.

The construction of the theory is based on the notion of a measure on a prering of any abstract space X . No topological structure of the space X is required for the

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development of the theory. The notion of a prering represents an abstraction from the family of intervals.

These measures are readily available and it is not necessary to extend them beforehand for the development of the theory to a Lebesgue type measure on a sigma-ring. They generalize Lebesgue measures.

If (X, V, ν) represents such a measure space one can construct in a single step the spaces $L(\nu, R)$ of Lebesgue summable functions and the spaces $L(\nu, Y)$ of Bochner summable functions over the space X and to develop their properties as Banach spaces and to obtain the theory of Lebesgue and Bochner integrals.

The main tools in developing of the theory are some elementary properties of Banach spaces concerning the norm $\| \cdot \|$, some knowledge of calculus, and familiarity with the geometric series

$$\sum_{n \geq 0} x^n = 1/(1 - x) \quad \text{for all } x \in (0, 1).$$

We shall follow the approach of Bogdanowicz [4] and [5] with some modifications to construct a generalized Lebesgue-Bochner-Stieltjes integral of the form $\int u(f, d\mu)$ where u is a bilinear operator acting in the product of Banach spaces, f is a Bochner summable function, and μ a vector-valued measure.

If the operator u represents the multiplication by scalars that is $u(\lambda, y) = \lambda y$ for y in a Banach space Y and μ represents a Lebesgue measure on a sigma ring, then the integral $\int u(f, d\mu)$ reduces to the classical Bochner integral $\int f d\mu$ and when Y represents the space of reals then the integral reduces to the classical Lebesgue integral.

Every linear continuous operator from the space of Bochner summable functions into any Banach space is representable by means of such integral.

2. PRERINGS AND RINGS OF SETS

Assume that X is any abstract space and V some family of subsets of X that includes the empty set \emptyset . Denote by $S(V)$ the family of all sets of X that are disjoint unions of finite collections of sets from the family V . Clearly empty set \emptyset belongs to $S(V)$. Such a family $S(V)$ will be called the **family of simple sets** generated by the family V .

We shall say that such a family V forms a **prering** of the space X , if the following conditions are satisfied for any two sets from V : if $A_1, A_2 \in V$, then both the intersection $A_1 \cap A_2$ and set difference $A_1 \setminus A_2$ belong to the family $S(V)$ of simple sets.

The notion of a prering represents an abstraction from the family of bounded intervals in the space of real numbers R or from the family of rectangles in the space R^2 .

A family of sets V of the space X is called a **ring** if V is a prering such that $V = S(V)$, which is equivalent to the following conditions: if $A_1, A_2 \in V$, then $A_1 \cup A_2 \in V$ and $A_1 \setminus A_2 \in V$.

It is easy to prove that a family V forms a prering if and only if the family $S(V)$ of the simple sets forms a ring. Every ring (prering) V of a space X containing the space X itself is called an **algebra (pre-algebra)** of sets, respectively.

If the ring V is closed under countable unions it is called a **sigma ring** (σ -ring for short.) If the ring V is closed under countable intersections it is called a **delta ring** (δ -ring for short.) It follows from de Morgan law that δ -algebra and σ -algebra represent the same notion.

3. TENSOR PRODUCT OF PRERINGS

We shall need in the sequel the following notions. Let $F = \{A_i\}$ be a nonempty finite family of sets of the space X . A finite family $G = \{B_j\}$ of disjoint sets is called a **refinement** of the family F if every set of the family F can be written as disjoint union of some sets from the family G .

Given two abstract spaces X_1 and X_2 . Assume that V_1 is a collection of subsets of the space X_1 and V_2 a collection of subsets of the space X_2 .

By **tensor product** $V_1 \otimes V_2$ of the families V_1 and V_2 we shall understand the family of subsets of the Cartesian product $X_1 \times X_2$ defined by the formula

$$V_1 \otimes V_2 = \{A_1 \times A_2 : A_1 \in V_1, A_2 \in V_2\}.$$

Theorem 3.1 (Tensor product of prerings is a prering). *Assume that V_j represents a prering of a space X_j where $j = 1, 2$. Then the tensor product $V_1 \otimes V_2$ represents a prering of the space $X_1 \times X_2$.*

Proof. Notice that the following two properties of a family V of subsets of a space X are equivalent:

- The family V forms a prering.
- The empty set belongs to V and for every two sets $A, B \in V$ there exists a finite disjoint refinement of the pair $\{A, B\}$ in the family V , that is, there exists a finite collection $D = \{D_1, \dots, D_k\}$ of disjoint sets from V such that each of the two sets A, B can be represented as a union of some sets from the collection D .

Clearly the tensor product $V_1 \otimes V_2$ of the prerings contains the empty set. Now take any pair of sets $A, B \in V_1 \otimes V_2$. We have $A = A_1 \times A_2$ and $B = B_1 \times B_2$. If one of the sets A_1, A_2, B_1, B_2 is empty then the pair A, B forms its own refinement from V . So consider the case when all the sets A_1, A_2, B_1, B_2 are nonempty.

Let $C = \{C_j \in V_1 : j \in J\}$ be a refinement of the pair A_1, B_1 and

$$D = \{D_k \in V_2 : k \in K\}$$

a refinement of A_2, B_2 . We may assume that the refinements do not contain the empty set.

The collection of sets $C \otimes D$ forms a refinement of the pair A, B . Indeed each set of the pair A_1, B_1 can be uniquely represented as the union of sets from the refinement C . Similarly each set of the pair A_2, B_2 can be represented in a unique way as union of sets from the refinement D . Since the sets of the collection $C \otimes D$ are disjoint and nonempty, each set of the pair $A_1 \times A_2$ and $B_1 \times B_2$ can be uniquely represented as the union of the sets from $C \otimes D$. Thus $V = V_1 \otimes V_2$ is a prering. \square

4. FUNDAMENTAL THEOREM ON PRERINGS

The following theorem characterizes prerings of any abstract space X . It shows that prerings provide a natural sufficient structure to build the theory of the integrals and of the spaces of summable functions.

Theorem 4.1 (Fundamental theorem on prerings). *Let V be a non-empty family of subsets of an abstract space X . Then the following statements are equivalent*

- (1) *The family V of sets forms a prering.*
- (2) *Every finite family of sets $\{A_1, A_2, \dots, A_k\} \subset V$ has a finite refinement in the family V .*
- (3) *For every finite collection of linear spaces Y_1, Y_2, \dots, Y_n, W and any map*

$$u : Y_1 \times Y_2 \times \dots \times Y_n \mapsto W$$

preserving zero, that is such that $u(0, 0, \dots, 0) = 0$, we have that the relations

$$s_1 \in S(V, Y_1), s_2 \in S(V, Y_2), \dots, s_n \in S(V, Y_n)$$

imply $s \in S(V, W)$, where the function s is defined by the formula

$$s(x) = u(s_1(x), s_2(x), \dots, s_n(x)) \quad \text{for all } x \in X.$$

- (4) *The family $S(V)$ of simple sets generated by V forms a ring of sets.*

We will need the following lemma.

Lemma 4.2. *Let V be a prering of sets. If $B \in V$ and $C_1, C_2, \dots, C_k \in V$ then the set*

$$B \setminus (C_1 \cup C_2 \cup \dots \cup C_k)$$

is a simple set, that is it belongs to the family $S(V)$.

To prove the lemma use mathematical induction on k and the set identity

$$B \setminus (C_1 \cup C_2 \cup \dots \cup C_{k+1}) = (B \setminus (C_1 \cup C_2 \cup \dots \cup C_k)) \setminus C_{k+1}.$$

Proof. We shall prove the theorem using a circular argument.

1 \implies 2. Assume that the family V forms a prering. Consider any family

$$E = \{B_1, \dots, B_k\} \subset V.$$

We shall apply induction with respect to k . For $k = 1$ the statement (2) is satisfied. Assume that for every family such that $k \leq n$ there exists a finite refinement from the prering V . Consider a family

$$F = \{B_1, \dots, B_{n+1}\} \subset V.$$

From the inductive assumption the family

$$\{B_2, \dots, B_{n+1}\}$$

has a finite refinement

$$\{C_1, \dots, C_m\} \subset V.$$

Notice that the following family

$$H = \{C_j \cap B_1, C_j \setminus B_1, B_1 \setminus (C_1 \cup \dots \cup C_m) : j = 1, 2, \dots, m\}$$

forms a refinement of the family F . Using the definition of a prering and the lemma one can prove that there exists a family $G \subset V$ forming a finite refinement of the family H and consequently of the family F . Thus we have proved the implication $1 \implies 2$.

$2 \implies 3$. To prove this implication take

$$s_j \in S(V, Y_j) \quad \text{for } j = 1, 2, \dots, n$$

We may assume that each of the functions is of the form

$$(4.1) \quad s_j = y_{j,1}c_{A_{j,1}} + \dots + y_{j,m_j}c_{A_{j,m_j}} \quad \text{for } j = 1, 2, \dots, m_j$$

where each of the sets $A_{j,i}$ is non-empty and $y_{j,i} \neq 0$. Let $\{B_1, B_2, \dots, B_t\} \subset V$ be a finite refinement of the collection of sets

$$\{A_{j,i} : j = 1, \dots, n; i = 1, \dots, m_j\}.$$

We may assume that each set B_j is non-empty. It follows from formula (4.1) and from the definition of a refinement that each of the functions s_j is constant on each set B_i and is equal to 0 outside the union $B_1 \cup \dots \cup B_t$. Let

$$s_j(x) = z_{j,i} \quad \text{when } x \in B_i; i = 1, 2, \dots, t.$$

Let

$$w_i = u(z_{1,i}, z_{2,i}, \dots, z_{n,i}) \quad \text{for } i = 1, 2, \dots, t.$$

Then the composed functions $s(x)$

$$s(x) = u(s_1(x), s_2(x), \dots, s_n(x)) \quad \text{for all } x \in X$$

is of the form

$$s = w_1c_{B_1} + w_2c_{B_2} + \dots + w_tc_{B_t} \in S(V, W).$$

This completes the proof of the implication $2 \implies 3$.

$3 \implies 4$ Notice the equivalence

$$A \in S(V) \Leftrightarrow c_A \in S(V, R).$$

Let $Y_1 = Y_2 = R$ be the space of reals. Define

$$u(r_1, r_2) = r_1 + r_2 - r_1r_2 \quad \text{for all } r_1 \in Y_1, r_2 \in Y_2.$$

The function u preserves zero $u(0, 0) = 0$ and we have

$$c_{A \cup B}(x) = u(c_A(x), c_B(x)) = c_A(x) + c_B(x) - c_A(x)c_B(x) \quad \text{for all } x \in X.$$

Thus $c_{A \cup B} \in S(V, R)$ and so $A \cup B \in S(V)$. Thus the family $S(V)$ of simple sets is closed under the finite union operation.

To prove that it is closed under the difference of sets operation notice the identity

$$c_{A \setminus B} = c_A - c_A c_B.$$

Using the function v defined by

$$v(r_1, r_2) = r_1 - r_1r_2 \quad \text{for all } r_1 \in Y_1, r_2 \in Y_2$$

similarly as before we can derive that the collection $S(V)$ of simple sets is closed under the difference of sets operation. Thus the implication $3 \implies 4$ has been proved.

$4 \implies 1$. To prove this implication assume that $S(V)$, the collection of simple sets generated by the family V , forms a ring of sets. Thus $S(V)$ is closed under finite union and set difference operations, that is

$$A, B \in S(V) \implies A \cup B, A \setminus B \in S(V).$$

Now from the set identity

$$A \cap B = (A \cup B) \setminus [(A \setminus B) \cup (B \setminus A)]$$

follows that the family $S(V)$ is closed under the operation of intersection of two sets.

Since $V \subset S(V)$ the above properties imply

$$A, B \in V \implies A \cap B \in S(V) \quad \text{and} \quad A \setminus B \in S(V).$$

Thus the family V forms a prering. This completes the proof of the implication $4 \implies 1$. Thus the theorem has been proved. \square

Corollary 4.3 (Case of a pre-algebra). *Let V be a non-empty family of subsets of an abstract space X . Then the following statements are equivalent*

- (1) *The family V of sets forms a pre-algebra.*
- (2) *For every finite collection of linear spaces Y_1, Y_2, \dots, Y_n, W and any map u from the Cartesian product $Y_1 \times Y_2 \times \dots \times Y_n$ into the space W we have that the relations*

$$s_1 \in S(V, Y_1), s_2 \in S(V, Y_2), \dots, s_n \in S(V, Y_n)$$

imply $s \in S(V, W)$, where the function s is defined by the formula

$$s(x) = u(s_1(x), s_2(x), \dots, s_n(x)) \quad \text{for all } x \in X.$$

- (3) *The family $S(V)$ of simple sets generated by V forms an algebra of sets.*

Proof. The proof is obvious and we leave it to the reader. \square

5. MEASURE SPACE

A finite-valued function v from a prering V into $[0, \infty)$, the non-negative reals, satisfying the following implication

$$(5.1) \quad A = \bigcup_{t \in T} A_t \implies v(A) = \sum_{t \in T} v(A_t)$$

for every set $A \in V$, that can be decomposed into finite or countable collection

$$A_t \in V \quad (t \in T)$$

of disjoint sets, will be called a **σ -additive positive measure**.

It was called a **positive volume** in the earlier papers of Bogdanowicz [4], [5], and [8]. Notice that since by definition every prering V contains the empty set \emptyset , from countable additivity (5.1) follows that $v(\emptyset) = 0$.

By **Lebesgue measure** over an abstract space X we shall understand any set function v from a σ -ring V of the space X into the extended non-negative reals $[0, \infty]$, that satisfies the implication (5.1) and has value zero on the empty set $v(\emptyset) = 0$. We have to postulate this explicitly to avoid the case of a trivial measure that is identically equal to ∞ .

Definition 5.1 (Measure space). *A triple (X, V, v) , where X denotes an abstract space and V a prering of the space X and v a σ -additive non-negative finite-valued measure on the prering V , will be called a **measure space** or **positive measure space**.*

It is clear that every *finite Lebesgue measure* forms a positive measure in our sense, and in the case when it has infinite values by stripping it of infinities we obtain a positive measure in our sense.

The infinite valued measures in the theory of integration are like oversized tires on the wheels of a car. They provide some convenience but they are not necessary. This can be easily noticed in the theory of Bochner summable functions: There no natural infinities in Banach spaces and one has to discard all sets of infinite measure to construct the integral. We shall discuss such measures in detail in a later section.

6. EXAMPLES OF MEASURES ON PRERINGS

It is good to have a few examples of the measure spaces at hand. The first example corresponds to Dirac's δ function.

Example 1 (Dirac measure space). *Let X be any abstract set and V the family of all subsets of the space X . Let x_0 be a fixed point of X . Let $v_{x_0}(A) = 1$ if $x_0 \in A$ and $v_{x_0}(A) = 0$ otherwise. Since V forms a sigma ring the triple (X, V, v) forms in this case a Lebesgue measure space.*

Example 2 (Counting measure space). *Let X be any abstract set and V consisting of the empty set and of all singletons*

$$V = \{\emptyset, \{x\} : x \in X\}.$$

Let $v(A) = 1$ for all singleton sets $A = \{x\}$ and $v(\emptyset) = 0$. The triple (X, V, v) forms a measure space that is not a Lebesgue measure space.

Example 3 (Striped Lebesgue measure space). *Assume that M is a sigma-ring of subsets of X and μ is any Lebesgue measure on M finite valued or with infinite values. Let*

$$V = \{A \in M : \mu(A) < \infty\}.$$

Plainly V forms a prering. Then restricting μ to V yields a positive measure space $(X, V, \mu.)$

The most important measure space to the sequel is the following.

Proposition 6.1 (Riemann measure space). *Let R denote the space of reals and V the collection of all bounded intervals I open, closed, or half-open. If $a \leq b$ are the end points of an interval I let $v(I) = b - a$. Then the triple (R, V, v) forms a measure space. We shall call this space the **Riemann measure space**.*

Proof. The collection V of intervals forms a prering. Indeed the intersection of any two intervals is an interval or an empty set. But empty set can be represented as an open interval $(a, a) = \emptyset$. The set difference of two intervals is either the union of two disjoint intervals or a single interval or an empty set. Thus we have that for any two intervals $I_1, I_2 \in V$ we have $I_1 \cap I_2 \in S(V)$ and $I_1 \setminus I_2 \in S(V)$. This proves that V is a prering.

To prove countable additivity assume that we have a decomposition of an interval I with ends $a \leq b$ into disjoint countable collection $I_t (t \in T)$ of intervals with end points $a_t \leq b_t$, that is

$$(6.1) \quad I = \bigcup_{t \in T} I_t.$$

The case when interval I is empty or consists of a single point is obvious. So without loss of generality we may assume that the interval I has a positive length and that our index set $T = \{1, 2, 3, \dots\}$. Take any $\varepsilon > 0$ such that $2\varepsilon < v(I)$. Let $I^\varepsilon = [a + \varepsilon, b - \varepsilon]$ and $I_t^\varepsilon = (a_t - \varepsilon 2^{-t}, b_t + \varepsilon 2^{-t})$ for all $t \in T$.

The family $I_t^\varepsilon (t \in T)$ forms an open cover of the compact interval I^ε thus there exists a finite set $J \subset T$ of indexes such that

$$I^\varepsilon \subset \bigcup_{t \in J} I_t^\varepsilon.$$

The above implies

$$\begin{aligned} v(I) - 2\varepsilon &= v(I^\varepsilon) \leq \sum_{t \in J} v(I_t^\varepsilon) \leq \sum_{t \in T} v(I_t^\varepsilon) \\ &= \sum_{t \in T} (v(I_t) + 2^{-t+1}\varepsilon) = \sum_{t \in T} v(I_t) + 2\varepsilon. \end{aligned}$$

Passing to the limit in the above inequality when $\varepsilon \rightarrow 0$ we get

$$v(I) \leq \sum_{t \in T} v(I_t)$$

On the other hand from the relation (6.1) follows that for any finite set J of indexes we have

$$I \supset \bigcup_{t \in J} I_t \implies v(I) \geq \sum_{t \in J} v(I_t).$$

Since

$$\sup_J \sum_{t \in J} v(I_t) = \sum_{t \in T} v(I_t)$$

we get from the above relations that the set function v is countably additive and thus it forms a measure. \square

As will follow from the development of this theory the Riemann measure space generates the same space of summable functions and the integral as the classical Lebesgue measure over the reals. It is good to see a few more examples of measures related to this one.

Proposition 6.2 (Riemann measure space over R^n). *Let $X = R^n$ and V the collection of all n -dimensional cubes of the form*

$$I_1 \times \dots \times I_n$$

where I_j represent intervals in the space R of reals. If $a_j \leq b_j$ are the end points of an interval I_j let

$$v(I_1 \times \dots \times I_n) = (b_1 - a_1)(b_2 - a_1) \cdots (b_n - a_n).$$

Then the triple (R^n, V, v) forms a measure space. We shall call this space the Riemann measure space over R^n .

Proof. The proof is similar to the preceding one and we leave it to the reader. \square

Proposition 6.3 (Stieltjes measure space). *Let R denote the space of reals, and g a nondecreasing function from R into R , and D the set of discontinuity points of g . Let V denote the collection of all bounded intervals I open, closed, or half-open with end points $a, b \notin D$. If $a \leq b$ are the end points of an interval I let $v(I) = g(b) - g(a)$. Then the triple (R, V, v) forms a measure space.*

Proof. The proof is similar to the proof for Riemann measure and we leave it to the reader. \square

A nondecreasing left-side continuous function F from the extended closed interval $E = [-\infty, +\infty]$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$ is called a **probability distribution function**. Any measure space (X, V, v) over a prering V such that $X \in V$ and $v(X) = 1$ is called a **probability measure space**.

Proposition 6.4 (Probability distribution generates probability measure space). *Let F be a probability distribution on the extended reals E . Let V consists of all intervals of the form $[a, b)$ or $[a, \infty]$, where $a, b \in E$. If $I \in V$ let $v(I)$ denote the increment of the function on the interval I similarly as in the case of Stieltjes measure space.*

Then the triple (E, V, v) forms a probability measure space.

Proof. To prove this proposition notice that the space E can be considered as compact space and the proof can proceed similarly as in the case of the Riemann measure space. \square

In the case of topological spaces there are two natural prerings available to construct a measure space: The prering consisting of differences $G_1 \setminus G_2$ of open sets, and the prering consisting of differences $Q_1 \setminus Q_2$ of compact sets.

7. CONSTRUCTING NEW MEASURE SPACES FROM OLD

Given some measure spaces we shall review here several construction allowing one to obtain new measure spaces. Assume that we have available measure spaces (X_t, V_t, v_t) where the index t runs through a set T of any cardinality. We can construct a direct sum of the sets X_t ($t \in T$) by considering the disjoint union X of the sets X_t ($t \in T$). Thus each space $X_t \subset X$ and

$$X_t \cap X_s = \emptyset \quad \text{if} \quad t \neq s; \quad t, s \in T.$$

Therefore the union

$$V = \bigcup_{t \in T} V_t$$

consists of subsets of the space X . It is evident that the family V forms a prering and the measure v on it is uniquely defined by the condition

$$v(A) = v_t(A) \quad \text{if} \quad A \in V_t, \quad \text{for some} \quad t \in T.$$

This measure space (X, V, v) will be called a **direct sum of measure spaces** (X_t, V_t, v_t) over $t \in T$.

Now consider the case when all the underling spaces $X_t = X$ coincide. Define

$$V = \left\{ A \subset X : A \in V_t \quad \forall t \in T, \quad v(A) = \sum_{t \in T} v_t(A) < \infty \right\}.$$

It is easy to prove that the triple (X, V, v) forms a measure space.

Another operation yielding a new measure space is the following. Consider a fixed measure space (X, V, v) and subfamily $V_0 \subset V$ forming a prering. Then the restriction (X, V_0, v) yields a measure space. A common example is the following. Take any set A in the prering V and let

$$V_A = \{B \in V : B \subset A\}.$$

restriction (A, V_A, v) yields a measure space. Thus, for instance, the Riemann measure space restricted to to any interval in R yields a measure over a pre-algebra.

Another important case is the following. Consider a pair of measure spaces (X_i, V_i, v_i) for $i = 1, 2$. Let $X = X_1 \times X_2$ denote the Cartesian product and $V = V_1 \otimes V_2$ the tensor product of the prerings. Let

$$v(A) = v_1(A_1)v_2(A_2) \quad \text{for all} \quad A = A_1 \times A_2 \in V.$$

For shorthand we shall use the notation $v_1 \otimes v_2$ to denote the set function v . Then the triple

$$(X, V, v) = (X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2)$$

forms a measure space called the **tensor product** of the measure spaces. This fact follows from the *Monotone Convergence Theorem*, also known as Beppo Levi's theorem, and we shall establish it in a later section.

8. MATHEMATICAL PRELIMINARIES

This section contains review of the essential notions concerning construction of Banach spaces [1] that are needed for understanding the development of the theory presented here.

A reader familiar with these notions can skip this section at the first reading and to return to it later as the need arises.

8.1. Construction of Basic Banach spaces. Let F denote either the Galois field R of reals or the Galois field C of complex numbers. We shall assume that the reader is familiar with the notion a **linear space** also called a **vector space**. In short in a vector space X there are given two operations: addition of vectors $x + y$ for any x, y , and multiplication by scalars λx for any $\lambda \in F$ and $x \in X$. The basic examples of vector spaces are

- Space R of reals considered as a vector space over the field R .
- Space C of complex numbers considered as a vector space over the field R .
- Space C of complex numbers considered as a vector space over the field C .
- More generally the space R^n or C^n for $n = 1, 2, \dots$
- The space of continuous functions from an interval I into a space R^n . This space will be denoted by $C(I, R^n)$.

We shall need in this paper methods of constructing Banach spaces from the basic spaces.

8.2. Topological space.

Definition 8.1. Let X be an abstract space. Assume that in the space X there is given a collection of subsets G having the following properties

- The empty set \emptyset and the entire space X belong to G .
- For any finite number of sets $A_i \in G$, ($i = 1, \dots, k$) their intersection $\bigcap_{i \leq k} A_i$ belongs to G .
- For any finite or infinite number of sets $A_i \in G$, where $i \in T$, their union $\bigcup_{i \in T} A_i$ belongs to G .

Then the pair (X, G) is called a **topological space**. The sets belonging to G are called **open** sets and the sets, whose complements $X \setminus A$ are open, are called **closed** sets. By a **neighborhood** of a point x one understands any open set containing the point x .

From de Morgan laws one can obtain the following characteristic properties of closed sets.

Corollary 8.2 (Closed sets). Denote by F the collection of all closed sets of a topological space X . Then the following is true

- The empty set \emptyset and the entire space X belong to F .
- For any finite number of sets $A_i \in F$, where $i = 1, \dots, k$, their union $\bigcup_{i \leq k} A_i$ belongs to F .
- For any finite or infinite number of sets $A_i \in F$, $i \in T$, their intersection $\bigcap_{i \in T} A_i$ belongs to F .

8.3. Seminormed and normed spaces.

Definition 8.3 (Seminorm, extended seminorm, norm). Let $R^+ = [0, \infty)$ denote the set of non-negative reals and $E^+ = [0, \infty]$ the set of extended non-negative reals. A functional p from a linear space X into R^+ or into E^+ is called a **seminorm** or an **extended seminorm**, respectively, if

$$p(x + y) \leq p(x) + p(y), \quad p(ax) = |a|p(x) \quad \text{for all } x, y \in X, a \in F.$$

The first of the above conditions is called the **triangle inequality** and the second the **homogeneity** condition. Such a functional is called a **norm** if in addition $p(x) = 0$ if and only if $x = 0$. Such a functional is called *nontrivial* if $0 < p(x) < \infty$ at least for one point $x \in X$.

Proposition 8.4 (Seminorm inequalities). If p is a seminorm defined on a vector space X , then the following inequalities are true

$$|p(x) - p(y)| \leq p(x - y) \quad \text{for all } x, y \in X$$

and

$$p(x - y) \geq p(x) - p(y) \quad \text{for all } x, y \in X.$$

Proof. The proof follows from the definition of a seminorm in particular from the triangle inequality $p(x + y) \leq p(x) + p(y)$. \square

A pair (X, p) where X is a vector space and p a finite-valued seminorm on X forms a topological space if open sets are defined by the following condition.

A set A is open if it has the following property, for every element $x \in A$ there exists a number $r > 0$ such that every point y with the property $p(y - x) < r$ belongs to A .

For the empty set this condition is satisfied in the vacuum. It is easy to check that indeed such a family of sets satisfies the axioms for topology.

We shall call such a pair (X, p) a **seminormed space**. If the functional p forms a norm, then the pair (X, p) is called a **normed space**.

In this case we use a more convenient notation

$$\|x\| = p(x) \quad \text{for all } x \in X.$$

The basic vector spaces R and C with the absolute value $|\cdot|$ form a normed-space. The spaces R^n and C^n form normed spaces with norm defined by the usual formula

$$\|(x_1, x_2, \dots, x_n)\| = \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} \quad \text{for all } x = (x_1, x_2, \dots, x_n) \in R^n \text{ or } C^n.$$

In the space $C(I, R^n)$ we define the norm by

$$\|f\| = \sup \{|f(t)| : t \in I\} \quad \text{for all } f \in C(I, R^n).$$

These spaces are all normed spaces. To see some example of a seminormed space that is not a normed space consider take R^2 and let

$$p(x_1, x_2) = |x_1| \quad \text{for all } (x_1, x_2) \in R^2.$$

8.4. Lipschitz condition. We use function as synonym with functional, map, operator, or transformation.

Definition 8.5. Assume that (X, p) and (Y, q) are two semi-normed spaces. A function f from a set $W \subset X$ into the space Y is said to be **Lipschitzian** on the set W if for some constant M we have

$$q(f(x_1) - f(x_2)) \leq M p(x_1 - x_2) \quad \text{for all } x_1, x_2 \in W.$$

Definition 8.6 (Continuity). A function f , from a topological space X into a topological space Y is said to be **continuous** if for every open set A in the space Y the set

$$(8.1) \quad B = \{x \in X : f(x) \in A\}$$

is open in the topological space X . The relation between the sets A and B given in (8.1) we write as $B = f^{-1}(A)$ and in such a case the set B is called the **inverse image** of the set A .

It is easy to prove that every Lipschitzian function is continuous. In particular every seminorm p is continuous when considered as a function from the seminormed space (X, p) into the space R of reals.

Using the above definition one can prove that a function f is continuous if and only if the inverse image $f^{-1}(F)$ of every closed set F is closed.

8.5. Convergence of a sequence or a series. In a semi-normed vector space (X, p) we introduce the notion of a sequential convergence.

Definition 8.7. We say that a sequence $x_n \in X$ **converges** to a point $x \in X$ if and only if

$$p(x - x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The limit point x in a semi-normed space is unique if and only if the functional p forms a norm.

By a **series** with the terms $x_n \in X$ we understand the sequence

$$s_n = \sum_{j \leq n} x_j.$$

A series x_n is said to be **convergent absolutely** with respect to the seminorm p if

$$\sum_{n=1}^{\infty} p(x_n) < \infty.$$

In a semi-normed space we have a convenient characterization of closed sets. A set F in such a space is closed if and only if for every convergent sequence x_n with terms in F the limit x of the sequence also is in F , that is the following implication is true

$$x_n \in F \quad \text{for all } n \quad \text{and} \quad x_n \rightarrow x \implies x \in F.$$

This gives us another characterization of continuous functions in such spaces: A function f from a set W in a semi-normed space X into a semi-normed space Y

is continuous if and only if for every sequence $x_n \in W$ that converges to a point $x \in W$ we must have that the values $f(x_n)$ converge to the value $f(x)$.

Another important property of semi-normed spaces is the following. For every absolutely convergent series with terms x_n that converges to a point $x \in X$ we have the estimate

$$p(x - \sum_{j=1}^n x_j) \leq \sum_{j=n+1}^{\infty} p(x_j) \quad \text{for all } n = 1, 2, \dots$$

8.6. Banach spaces and complete seminormed spaces.

Definition 8.8. A normed space $(X, \|\cdot\|)$, in which every absolutely convergent series x_n is convergent to some point $x \in X$, is called a **Banach space**.

In the theory of integration it is convenient to consider semi-normed spaces that are not normed but still every absolutely convergent series is convergent to a point.

Such spaces will be called in the sequel the **complete semi-normed spaces** and the corresponding seminorms, **complete seminorms**.

Notice that the basic vector spaces that we introduced so far are Banach spaces.

Definition 8.9 (Lower semi-continuity). If f is a function from a topological space X into the extended reals $E = [-\infty, \infty]$ then we say that it is lower semi-continuous if for every finite number $a \in R$ the set

$$\{x \in X : f(x) > a\} = \{x \in X : f(x) \in (a, \infty]\}$$

is open or equivalently, by taking the complements of the sets, that the set

$$\{x \in X : f(x) \leq a\} = \{x \in X : f(x) \in [-\infty, a]\}$$

is closed.

Notice that every continuous function is lower semi-continuous.

Proposition 8.10. Assume that $(X, \|\cdot\|)$ is a normed vector space and $p_t (t \in T)$ a family of extended seminorms from X into $E^+ = [0, \infty]$. If each seminorm p_t is lower semi-continuous then

$$p(x) = \sup \{p_t(x) : t \in T\} \quad \text{for all } x \in X$$

represents a lower semi-continuous extended seminorm on X .

Proof. It is plain that p forms a seminorm. For any $a \in R$ notice the identity

$$\{x \in X : p(x) \leq a\} = \bigcap_{t \in T} \{x \in X : p_t(x) \leq a\}.$$

Thus the set on the left is closed as an intersection of a family of closed sets. Hence the seminorm p is lower semi-continuous. \square

Theorem 8.11 (Generating Banach space X_p). *Assume that $(X, \| \cdot \|)$ represents a Banach space and $p : X \rightarrow E^+$ is a lower semi-continuous extended semi-norm. Let*

$$X_p = \{x \in X : p(x) < \infty\} \quad \text{and} \quad \|x\|_p = \|x\| + p(x) \quad \text{for } x \in X_p$$

Then the pair $(X_p, \| \cdot \|_p)$ forms a Banach space.

Proof. It is easy to see that the space X_p is linear and $\| \cdot \|_p$ forms a norm on it. We need to prove completeness of the norm. Take any series $x_n \in X_p$ that is absolutely convergent $\sum_n \|x_n\|_p < \infty$ and let a be a number such that

$$\sum_n \|x_n\|_p < a < \infty.$$

Since $\|x_n\| \leq \|x_n\|_p$ and $p(x_n) \leq \|x_n\|_p$ for all n we have the estimates

$$\sum_n \|x_n\| < a \quad \text{and} \quad \sum_n p(x_n) < a.$$

Since by assumption the norm $\| \cdot \|$ is complete, there exists a point $x \in X$ such that

$$(8.2) \quad \left\| x - \sum_{j \leq n} x_j \right\| \leq \sum_{j > n} \|x_j\|.$$

Let us introduce notation

$$B_r = \{x \in X : p(x) \leq r\} \quad \text{for all } r \geq 0.$$

Notice that $B_r \subset X_p$ and that from lower semi-continuity of p follows that the sets B_r are closed. Let

$$r_n = \sum_{n < j} p(x_j) \quad \text{for all } n = 1, 2, \dots$$

Since

$$p\left(\sum_{n < j \leq m} x_j\right) \leq \sum_{n < j \leq m} p(x_j) \leq \sum_{n < j} p(x_j) = r_n,$$

we have

$$\sum_{n < j \leq m} x_j \in B_{r_n} \quad \text{for all } m > n \text{ and } n = 1, 2, \dots$$

For fixed n define the sequence y_m by

$$y_m = \sum_{n < j \leq m} x_j \quad \text{for all } m > n \quad \text{and} \quad y_m = 0 \quad \text{otherwise.}$$

Notice that when $m \rightarrow \infty$, then

$$y_m = \sum_{n < j \leq m} x_j = \sum_{j \leq m} x_j - \sum_{j \leq n} x_j \rightarrow \sum_{j=1}^{\infty} x_j - \sum_{j \leq n} x_j,$$

where the convergence is in the norm $\| \cdot \|$. Since $y_m \in B_{r_n}$ for all m , and by lower semi-continuity of p the set B_{r_n} is closed, we must have

$$x - \sum_{j \leq n} x_j = \sum_{j=1}^{\infty} x_j - \sum_{j \leq n} x_j \in B_{r_n} \quad \text{for all } n,$$

that is

$$p\left(x - \sum_{j \leq n} x_j\right) \leq r_n = \sum_{n < j} p(x_j) \quad \text{for all } n = 1, 2, \dots$$

Since all $x_j \in X_p$ the above implies that $x \in X_p$ as follows from the triangle inequality. Combining the above with (8.2) we get

$$\begin{aligned} \left\| x - \sum_{j \leq n} x_j \right\|_p &= \left\| x - \sum_{j \leq n} x_j \right\| + p(x - \sum_{j \leq n} x_j) \\ &\leq \sum_{n < j} \|x_j\| + \sum_{n < j} p(x_j) = \sum_{n < j} \|x_j\|_p \quad \text{for all } n \end{aligned}$$

Thus we have proved that every absolutely convergent series in the normed space $(X_p, \|\cdot\|_p)$ converges to some point $x \in X_p$. Hence the pair $(X_p, \|\cdot\|_p)$ forms a Banach space. \square

Theorem 8.12 (Space $C_\delta(J, Y)$). *Assume that J is any interval, bounded or unbounded, closed or open, or partially closed, in the space R of reals and Y a Banach space.*

Assume that δ is a non-negative continuous function from J into R . Assume that $C_\delta(J, Y)$ denotes the set of all continuous functions from the interval J into the Banach space Y such that

$$|f(x)| \leq M\delta(x) \quad \text{for all } x \in J \quad \text{and some } M.$$

Let $\|f\|$ denote the infimum, that is the greatest lower bound, of all constants M appearing in the above condition.

Then the functional $\|\cdot\|$ forms a norm and the set $C_\delta(J, Y)$ equipped with the norm $\|\cdot\|$ forms a Banach space.

Proof. Let us introduce a shorthand notation $C_\delta = C_\delta(J, Y)$. It is clear that the set C_δ is non-empty since it contains for instance all functions of the form $\delta(x)y$ where y is any element of Y . It follows from the definition of the infimum of a set that

$$(8.3) \quad |f(x)| \leq \delta(x) \|f\| \quad \text{for all } f \in C_\delta, x \in J.$$

Let us prove that C_δ forms a linear space and $\|\cdot\|$ forms a norm.

To this end take any scalars λ_1, λ_2 and functions $f_1, f_2 \in C_\delta$. We have

$$\begin{aligned} |\lambda_1 f_1(x) + \lambda_2 f_2(x)| &\leq |\lambda_1| |f_1(x)| + |\lambda_2| |f_2(x)| \\ &\leq |\lambda_1| \delta(x) \|f_1\| + |\lambda_2| \delta(x) \|f_2\| \\ &\leq \delta(x) (|\lambda_1| \|f_1\| + |\lambda_2| \|f_2\|). \end{aligned}$$

Thus from above we get $\lambda_1 f_1 + \lambda_2 f_2 \in C_\delta$, so C_δ is linear, and

$$\|\lambda_1 f_1 + \lambda_2 f_2\| \leq |\lambda_1| \|f_1\| + |\lambda_2| \|f_2\| \quad \text{for all } \lambda_1, \lambda_2 \text{ and } f_1, f_2 \in C_\delta.$$

The above inequality implies that $\|\cdot\|$ forms a seminorm. Thus from the estimate (8.3) follows that the seminorm is zero if and only if the function f is identically zero. Thus $\|\cdot\|$ is a norm.

To prove completeness take any absolutely convergent series $f_n \in C_\delta$. We have

$$\sum_n \|f_n\| < \infty$$

and

$$|f_n(x)| \leq \delta(x) \|f_n\| \quad \text{for all } x \in J, n = 1, 2, \dots$$

Since the function δ as a continuous function is bounded on every bounded subinterval of J , the series

$$\sum_n f_n(x) = f(x)$$

converges uniformly and absolutely to some continuous function $f(x)$ on such subinterval. Thus the limit function f is continuous.

Since

$$|f(x)| \leq \sum_n |f_n(x)| \leq \delta(x) \sum_n \|f_n\| \quad \text{for all } x \in J,$$

the function f belongs to the set C_δ . Thus we have

$$\begin{aligned} |f(x) - \sum_{j \leq n} f_j(x)| &\leq \delta(x) \left\| f - \sum_{j \leq n} f_j \right\| \\ &\leq \delta(x) \sum_{j > n} \|f_j\| \quad \text{for all } x \in J, n = 1, 2, \dots \end{aligned}$$

Therefore by definition of the norm $\| \cdot \|$ we have

$$\left\| f - \sum_{j \leq n} f_j \right\| \leq \sum_{j > n} \|f_j\|.$$

Thus the space C_δ forms a Banach space. \square

In the above theorem we did not assume about the function δ that it is bounded or that it does not take on the value zero at some points. When $\delta(x) = 1$ for all $x \in J$, we get that the space C_δ coincides with the set of all continuous bounded functions on the interval J and C_δ forms a Banach space. We will need a more general theorem for the case of continuous bounded functions.

Theorem 8.13 (Space $C(W, Y)$). *Assume that W is a topological space and Y a Banach space. Assume that $C(W, Y)$ denotes the set of all continuous bounded functions from the space W into the Banach space Y . Let*

$$\|f\| = \sup \{|f(x)| : x \in W\} \quad \text{for all } f \in C(W, Y).$$

Then the set $C(W, Y)$ equipped with the above norm forms a Banach space.

Proof. The proof is similar to the previous one and we leave it to the reader. \square

9. LINEARITY AND BILINEARITY IN BANACH SPACES

Assume that Y, Z, W represent some Banach spaces either over the field R of reals or over the field C of complex numbers. By linear operator T , say from the space Y into the space W , we shall understand any **additive operator**, that is such that

$$T(y_1 + y_2) = T(y_1) + T(y_2) \quad \text{for all } y_1, y_2 \in Y,$$

and moreover it is either **homogeneous**

$$T(\lambda y) = \lambda T(y) \quad \text{for all } y \in Y \text{ and } \lambda \in C$$

or **conjugate homogeneous**

$$T(\lambda y) = \bar{\lambda} T(y) \quad \text{for all } y \in Y \text{ and } \lambda \in C.$$

In the case of Banach spaces over the field C of complex numbers every linear operator with respect to complex scalars is at the same time a linear operators with respect to the real scalars.

By a bilinear operator u from the product $Y \times Z$ into the space W we shall understand an operator such that the operators

$$y \mapsto u(y, z) \quad \text{and} \quad z \mapsto u(y, z)$$

are linear when the other variable is fixed.

As an example of such an operator consider Dirac's bra-ket $\langle y|z\rangle$ operator. In this case Y is a Hilbert space over the field of complex numbers and the operator is given by the formula

$$u(y, z) = \langle y|z\rangle \quad \text{for all } y, z \in Y,$$

where u is from $Y \times Y$ into C , and represents an inner product operator. Such operator is additive in each variable and homogeneous in variable z and conjugate homogeneous in the variable y .

We shall require the operator u to be **bounded** that is to satisfy the following condition

$$|u(y, z)| \leq m|y||z| \quad \text{for all } y \in Y, z \in Z \text{ and some } m \in R.$$

The set of all such bilinear operators with the same type of homogeneity with respect to complex numbers with usual operations of addition of functions and multiplication of a function by a number forms a Banach space with norm defined as $|u| = \inf \{m\}$ where m runs through all constants satisfying the above condition.

Denote by U the space of all bilinear bounded operators u from the space $Y \times Z$ into W . Norms of elements in the spaces Y, Z, W, U will be denoted by $|\cdot|$.

10. CLASSICAL APPROACH TO LEBESGUE-BOCHNER INTEGRATION

The development of the classical Lebesgue-Bochner theory of the integral goes through the following main stages as in Halmos [18] and Dunford and Schwartz [12]:

- The construction and development of the Caratheodory theory of outer measure v^* over an abstract space X .
- The construction of the Lebesgue measure v on the sigma ring V of measurable sets of the space X induced by the outer measure v^* .
- The development of the theory of real-valued measurable functions $M(v, R)$.
- The construction of the Lebesgue integral $\int f dv$.
- The construction and development of the theory of the space $L(v, R)$ of Lebesgue summable functions.
- The construction and development of the theory of the space $M(v, Y)$ of Bochner measurable functions.
- The construction of the Bochner integral $\int f dv$ and of the space $L(v, Y)$ of Bochner summable functions f from the space X into any Banach space Y .

The construction of the classical Lebesgue integral is an abstraction from the area under the graph of the function similar to the ideas of Riemann though different in execution.

From the point of view of Functional Analysis both the Lebesgue and Bochner integrals are particular linear continuous operators from the space $L(v, Y)$ of Bochner summable functions into the Banach space Y . Moreover from the theory of the space $L(v, Y)$ one can easily derive the theory of the spaces $M(v, Y)$, $M(v, R)$, and $L(v, R)$ and of the Lebesgue and Bochner integrals and also the theory of Lebesgue measure. For details see Bogdanowicz [5] and [8].

We shall show in brief how one can develop the theory of the space $L(v, Y)$ and to construct an integral of the form $\int u(f, d\mu)$, where u is any bilinear bounded operator from the product $Y \times Z$ of Banach spaces into a Banach space W and μ represents a vector measure. This integral for the case, when the spaces Y, Z, W are equal to the space R of reals and the bilinear operator u represents multiplication $u(y, z) = yz$, coincides with the Lebesgue integral

$$(10.1) \quad \int f dv = \int u(f, dv) \quad \text{for all } f \in L(v, R).$$

In the case, when $Y = W$ and $Z = R$ and $u(y, z) = zy$ represents the scalar multiplication, the integral coincides with the Bochner integral

$$(10.2) \quad \int f dv = \int u(f, dv) \quad \text{for all } f \in L(v, Y).$$

11. VECTOR MEASURES

Definition 11.1 (Vector measure space). *Assume that (X, V, v) represents a positive measure space over the prering V . A set function μ from a prering V into a Banach space Z is called a **vector measure** if for every finite family of disjoint sets $A_t \in V (t \in T)$ the following implication is true*

$$(11.1) \quad A = \bigcup_T A_t \in V \implies \mu(A) = \sum_T \mu(A_t).$$

Denote by $K(v, Z)$ the space of all vector measures μ from the prering V into the space Z , such that

$$|\mu(A)| \leq m v(A) \quad \text{for all } A \in V \text{ and some } m.$$

The least constant m satisfying the above inequality is denoted by $\|\mu\|$. It is easy to see that the pair $(K(v, Z), \|\mu\|)$ forms a Banach space.

12. CONSTRUCTION OF THE ELEMENTARY INTEGRAL SPACES

Assume that c_A denotes the characteristic function of the set A that is $c_A(x) = 1$ on A and takes value zero elsewhere. Let $S(V, Y)$ denote the space of all functions of the form

$$(12.1) \quad h = y_1 c_{A_1} + \dots + y_k c_{A_k}, \text{ where } y_i \in Y, A_i \in V$$

and the sets A_i in above formula are disjoint.

Notice that we extended the multiplication by scalars by agreement $y\lambda = \lambda y$ for all vectors y and scalars λ . The family $S(V, Y)$ of functions will be called the family of **simple functions** generated by the prering V . For fixed $u \in U$ and $\mu \in K(v, Z)$ and any simple function $h \in S(V, Y)$ define the operator

$$\int u(h, d\mu) = u(y_1, \mu(A_1)) + \dots + u(y_k, \mu(A_k))$$

and the integral operator

$$\int h dv = y_1 v(A_1) + \dots + y_k v(A_k).$$

The operators $\int h dv$ and $\int u(h, d\mu)$ are well defined, that is, they do not depend on the representation of the function h in the form (12.1). To prove this use the fact that any finite collection of sets from a prering has a finite refinement.

Let $|h|$ denote the function defined by the formula $|h|(x) = |h(x)|$ for $x \in X$. We see that if $h \in S(V, Y)$, then $|h| \in S(V, R)$. Therefore the following functional $\|h\| = \int |h| dv$ is well defined for all $h \in S(V, Y)$.

The following development of the theory of Lebesgue and Bochner summable functions and of the integrals are from Bogdanowicz [4].

Lemma 12.1 (Elementary integrals on simple functions). *The following statements describe the basic relations between the notions that we have just introduced.*

- (1) *The space $S(V, Y)$ is linear, $\|h\|$ is a seminorm on it, and $\int h dv$ is a linear operator on $S(V, Y)$, and moreover*

$$\left| \int h dv \right| \leq \|h\| \quad \text{for all } h \in S(V, Y).$$

- (2) *If $g \in S(V, R)$ and $f \in S(V, Y)$, then $gf \in S(V, Y)$.*
- (3) *$\int h dv \geq 0$ if $h \in S(V, R)$ and $h(x) \geq 0$ for all $x \in X$.*
- (4) *$\int g dv \geq \int f dv$ if $g, f \in S(V, R)$ and $g(x) \geq f(x)$ for all $x \in X$.*
- (5) *The operator $\int u(h, d\mu)$ is trilinear from the product space*

$$U \times S(V, Y) \times K(v, Z)$$

into the space W and

$$\left| \int u(h, d\mu) \right| \leq |u| \|h\| \|\mu\|$$

for all

$$u \in U, h \in S(V, Y), \mu \in K(v, Z).$$

Proof. The proof of the lemma is obvious and we leave it to the reader. \square

13. NULL SETS

Assume that (X, V, v) represents a positive measure space over the pre-ring V of an abstract space X .

Definition 13.1 (Null sets). *Let N be the family of all sets $A \subset X$ such that for every $\varepsilon > 0$ there exists a countable family $A_t \in V (t \in T)$ such that*

$$A \subset \bigcup_T A_t \quad \text{and} \quad \sum_T v(A_t) < \varepsilon.$$

*Sets of the family N will be called **null-sets** with respect to the measure v .*

The sets of the family N play the role of the sets of Lebesgue measure zero and in fact this is how Lebesgue himself defined this family in the case of sets in the space R of reals.

This family represents a **sigma-ideal** of sets in the power set $\mathcal{P}(X)$, that is, it has the following properties: if $A \in N$, then $B \cap A \in N$ for any set $B \subset X$, and the union of any countable family of null-sets $A_t \in N (t \in T)$ is also a null-set $\bigcup_T A_t \in N$.

To see this fact take any $\varepsilon > 0$ and assume that $T = \{1, 2, \dots\}$ represents the set of natural numbers. It is sufficient to cover each set A_t with a countable collection Q_t of sets from the pre-ring V with total sum of measures that does not exceed

$$2^{-t}\varepsilon \quad \text{for all } t = 1, 2, \dots$$

Clearly the collection

$$Q = \bigcup_T Q_t$$

is countable, it covers the set $A = \bigcup_T A_t$, and the total sum of the measures of sets in this collection does not exceed ε .

A condition $C(x)$ depending on a parameter $x \in X$ is said to be **satisfied almost everywhere** if there exists a null-set $A \in N$ such that the condition is satisfied at every point of the set $X \setminus A$.

14. NULL SETS IN CASE OF RIEMANN MEASURE SPACE

To see some examples of null sets consider the Riemann measure space (R, V, v) over the reals R . Clearly the empty set $\emptyset = (a, a)$, and any singleton $[b, b]$ is a null. Moreover any countable set of points forms a null set.

There exist null sets that are uncountable. A typical example of such a set is **Cantor's set**.

To construct Cantor's set take the closed interval $[0, 1]$ and divide it into three equal intervals. From the middle remove the open interval $(1/3, 2/3)$. The remaining two closed intervals have total length $2/3$ and they form a closed set F_1 . Repeat this process with each of the remaining intervals.

After n -steps the remaining set F_n will consist of the union of 2^n disjoint closed intervals of total length of $(2/3)^n$. The sets F_n are nested and their intersection $F = \bigcap_n F_n$ will represent a nonempty closed set called the Cantor set.

Cantor's set is a null set of cardinality equal to cardinality of the interval $[0, 1]$. Indeed, the set F can be covered by a countable number of intervals of total length as small as we please. Notice that a finite cover by intervals we can always augment by a sequence of intervals of the form (a, a) , that is by empty sets to get a countable cover.

To prove that the set F is of the same cardinality as the interval $[0, 1]$ consider expansions into infinite fractions at the base 3 of points belonging to F . Notice that the expansion must be of the form

$$x = 0.d_1, d_2, d_3, \dots$$

where the digits $d_i \in \{0, 2\}$. Ignore the set of points which have periodic expansions since they represent some rational numbers that form a countable set.

Similarly consider the binary expansions, that is at base 2, into non-periodic sequence of digits of points y of the set $[0, 1]$. We have

$$y = 0.a_1, a_2, a_3, \dots$$

where $a_i \in \{0, 1\}$.

Clearly the map $x \mapsto y$ given by the formula

$$a_i = d_i/2 \quad \text{for all } i = 1, 2, \dots$$

is one-to-one and onto. Thus the cardinalities of F and $[0, 1]$ are equal. Hence Cantor's set F is a null set of cardinality of a continuum.

15. FUNDAMENTAL LEMMAS

Definition 15.1 (Basic sequence). *By a **basic sequence** we shall understand a sequence $s_n \in S(V, Y)$ of simple functions for which there exists a series with terms $h_n \in S(V, Y)$ and a constant $M > 0$ such that $s_n = h_1 + h_2 + \dots + h_n$, where $\|h_n\| \leq M4^{-n}$ for all $n = 1, 2, \dots$*

The idea of the basic sequence as a series of simple functions with geometric rate of convergence can be traced to the work of Riesz. See [24], page 59, the proof of Riesz-Fisher theorem. The proof of Egoroff's theorem [13] provided the idea for the rest of the needed structure. Thus it seems appropriate to name the next lemma as Riesz-Egoroff property of a basic sequence.

Lemma 15.2 (Riesz-Egoroff property of a basic sequence). *Assume that (X, V, v) is a positive measure space on a pre-ring V and Y is a Banach space. Then the following is true.*

- (1) [Riesz] *If $s_n \in S(V, Y)$ is a basic sequence, then there exists a function f from the set X into the Banach space Y and a null-set A such that $s_n(x) \rightarrow f(x)$ for all $x \in X \setminus A$.*
- (2) [Egoroff] *Moreover, for every $\varepsilon > 0$ and $\eta > 0$, there exists an index k and a countable family of sets $A_t \in V (t \in T)$ such that*

$$A \subset \bigcup_T A_t \quad \text{and} \quad \sum_T v(A_t) < \eta$$

and for every $n \geq k$

$$|s_n(x) - f(x)| < \varepsilon \quad \text{if} \quad x \notin \bigcup_T A_t.$$

Proof. For proof of this lemma see Lemma 2 of Bogdanowicz [4, page 493]. \square

The idea for the following lemma came from Dunford-Schwartz [12], Part I, p. 111, Lemma 16.

Lemma 15.3 (Dunford's Lemma). *Assume that (X, V, v) is a positive measure space on a prepring V and Y is a Banach space. Then the following is true.*

If $s_n \in S(V, Y)$ is a basic sequence converging almost everywhere to zero 0, then the sequence of seminorms $\|s_n\|$ converges to zero.

Proof. For proof of this lemma see Lemma 3 of Bogdanowicz [4, pages 493-495]. \square

16. THE SPACES OF LEBESGUE AND BOCHNER SUMMABLE FUNCTIONS

Definition 16.1 (Lebesgue and Bochner spaces). *Assume that (X, V, v) is a measure space over a prepring V of an abstract space X and Y is a Banach space.*

Let $L(v, Y)$ denote the set of all functions $f : X \rightarrow Y$, such that there exists basic sequence $s_n \in S(V, Y)$ that converges almost everywhere to the function f .

*The space $L(v, Y)$ is called the space of **Bochner summable** functions and, for the case when Y is equal to the space R of reals, $L(v, R)$ represents the space of **Lebesgue summable** functions.*

Define

$$\|f\| = \lim_n \|s_n\|, \int u(f, d\mu) = \lim_n \int u(s_n, d\mu), \int f dv = \lim_n \int s_n dv.$$

Since the difference of two basic sequences is again a basic sequence, therefore it follows from the Elementary Lemma 12.1 and Dunford's Lemma 15.3 that the operators are well defined, that is, their values do not depend on the choice of the particular basic sequence convergent to the function f .

Theorem 16.2 (Basic properties of the space $L(v, Y)$). *Assume that (X, V, v) is a positive measure space on a prepring V and Y is a Banach space. Then the following is true.*

- (1) *The space $L(v, Y)$ is linear and $\|f\|$ represents a seminorm being an extension of the seminorm from the space $S(V, Y)$ of simple functions.*
- (2) *We have $\|f\| = 0$ if and only if $f(x) = 0$ almost everywhere.*
- (3) *The functional $\|f\|$ is a complete seminorm on $L(v, Y)$ that is given a sequence of functions $f_n \in L(v, Y)$ such that $\|f_n - f_m\| \xrightarrow{nm} 0$ there exists a function $f \in L(v, Y)$ such that $\|f_n - f\| \xrightarrow{n} 0$.*
- (4) *If $f_1(x) = f_2(x)$ almost everywhere and $f_2 \in L(v, Y)$, then $f_1 \in L(v, Y)$ and*

$$\|f_1\| = \|f_2\|, \int f_1 dv = \int f_2 dv, \int u(f_1, d\mu) = \int u(f_2, d\mu).$$

- (5) *The operator $\int f dv$ is linear and represents an extension onto $L(v, Y)$ of the operator from $S(V, Y)$. It satisfies the condition $|\int f dv| \leq \|f\|$ for all $f \in L(v, Y)$.*

- (6) The operator $\int u(f, d\mu)$ is trilinear on $U \times L(v, Y) \times K(v, Z)$ and represents an extension of the operator from the space $U \times S(V, Y) \times K(v, Z)$. It satisfies the condition:

$$|\int u(f, d\mu)| \leq |u| \|f\| \|\mu\| \quad \text{for all } u \in U, f \in L(v, Y), \mu \in K(v, Z).$$

Proof. For proof of this theorem see Theorem 1 of Bogdanowicz [4, page 495]. \square

Lemma 16.3 (Density of simple functions in $L(v, Y)$). *Assume that (X, V, v) is a positive measure space on a prepring V and Y is a Banach space. Let $s_n \in S(V, Y)$ be a basic sequence convergent almost everywhere to a function f . Then $\|s_n - f\| \rightarrow 0$.*

Proof. For proof of this lemma see Lemma 4 of Bogdanowicz [4, page 495]. \square

From Theorem 16.2 we see that the obtained integrals are continuous under the convergence with respect to the seminorm $\|\cdot\|$, that is, if $\|f_n - f\| \rightarrow 0$, then

$$\int f_n dv \rightarrow \int f dv \quad \text{and} \quad \int u(f_n, d\mu) \rightarrow \int u(f, d\mu).$$

The following theorem characterizes convergence with respect to this seminorm.

Theorem 16.4 (Characterization of the seminorm convergence). *Assume that (X, V, v) is a positive measure space on a prepring V and Y is a Banach space.*

Assume that we have a sequence of summable functions $f_n \in L(v, Y)$ and some function f from the set X into the Banach space Y .

Then the following conditions are equivalent

- (1) *The sequence f_n is Cauchy, that is $\|f_n - f_m\| \xrightarrow{nm} 0$, and there exists a subsequence f_{k_n} convergent almost everywhere to the function f .*
- (2) *The function f belongs to the space $L(v, Y)$ and $\|f_n - f\| \rightarrow 0$.*

Proof. For proof of this theorem see Theorem 2 of Bogdanowicz [4, page 496]. \square

When the space $Y = \mathbb{R}$, the space $L(v, \mathbb{R})$ represents the space of Lebesgue summable functions. We have the following relation between Bochner summable functions and Lebesgue summable functions.

Theorem 16.5 (Norm of Bochner summable function is Lebesgue summable). *Let (X, V, v) be a positive measure space on a prepring V and assume that Y is a Banach space.*

If f belongs space $L(v, Y)$ of Bochner summable functions, then the function $|f|$ defined by the formula

$$|f|(x) = |f(x)| \quad \text{for all } x \in X$$

belongs to the space $L(v, \mathbb{R})$ of Lebesgue summable functions and we have the identity

$$\|f\| = \int |f| dv \quad \text{for all } f \in L(v, Y).$$

Proof. For proof of this theorem see Theorem 3 of Bogdanowicz [4, page 496]. \square

Theorem 16.6 (Properties of Lebesgue summable functions). *Let (X, V, v) be a positive measure space on a prerings V and $L(v, R)$ the Lebesgue space of v -summable functions.*

- (a): *If $f \in L(v, R)$ and $f(x) \geq 0$ almost everywhere on X then $\int f dv \geq 0$.*
- (b): *If $f, g \in L(v, R)$ and $f(x) \geq g(x)$ almost everywhere on X then $\int f dv \geq \int g dv$.*
- (c): *If $f, g \in L(v, R)$ and $h(x) = \sup\{f(x), g(x)\}$ for all $x \in X$ then $h \in L(v, R)$.*
- (d): *Let $f_n \in L(v, R)$ be a monotone sequence with respect to the relation less or equal almost everywhere. Then there exists a function $f \in L(v, R)$ such that $f_n(x) \rightarrow f(x)$ almost everywhere on X and $\|f_n - f\| \rightarrow 0$ if and only if the sequence of numbers $\int f_n dv$ is bounded.*
- (e): *Let $g, f_n \in L(v, R)$ and $f_n(x) \leq g(x)$ almost everywhere on X for $n = 1, 2, \dots$. Then the function $h(x) = \sup\{f_n(x) : n = 1, 2, \dots\}$ is well defined almost everywhere on X and is summable, that is, $h \in L(v, R)$. A function defined almost everywhere is said to be summable if it has a summable extension onto the space X .*

Proof. For proof of this theorem see Theorem 4 of Bogdanowicz [4, page 496-497]. \square

From part (d) of the above theorem we can get the classical theorem due to Beppo Levi [20].

Theorem 16.7 (Beppo Levi's Monotone Convergence Theorem). *Assume that (X, V, v) is a positive measure space on a prerings V and $L(v, R)$ the Lebesgue space of v -summable functions.*

Let $f_n \in L(v, R)$ be a monotone sequence with respect to the relation less or equal almost everywhere. Then there exists a function $f \in L(v, R)$ such that $f_n(x) \rightarrow f(x)$ almost everywhere on X and

$$\int f_n dv \rightarrow \int f dv$$

if and only if the sequence of numbers $\int f_n dv$ is bounded.

Beppo Levi's theorem has an equivalent formulation in terms of a series.

Theorem 16.8 (Beppo Levi's theorem for a series). *Assume that (X, V, v) is a positive measure space on a prerings V and $L(v, R)$ the Lebesgue space of v -summable functions. Let f_n be a sequence of nonnegative Lebesgue summable functions.*

Then there exists a function $f \in L(v, R)$ such that

$$\sum_n f_n(x) = f(x) \quad \text{almost everywhere on } X$$

and

$$\sum_n \int f_n dv = \int \sum_n f_n dv = \int f dv$$

if and only if the sum of the series

$$\sum_{n=1}^{\infty} \int f_n dv < \infty$$

is finite.

Definition 16.9 (Linear lattice). Assume that L represents a linear space of functions from an abstract space X into the reals R . If the space L is closed under the map $f \mapsto |f|$ then the operations $f \vee g$ and $f \wedge g$ given by the formula

$$(f \vee g)(x) = \sup \{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) \quad \text{and}$$

$$(f \wedge g)(x) = \inf \{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|) \quad \text{for all } x \in X$$

are well defined. Such a space L will be called a **linear lattice** of functions.

The operation $f \vee g$ is called the **meet** of functions f, g and the operation $f \wedge g$ the **joint**. Notice that meet operation is commutative: $f \wedge g = g \wedge f$, and associative

$$(f \wedge g) \wedge h = f \wedge (g \wedge h),$$

and so is the **joint** operation.

Notice that from Theorem (16.5) follows that absolute value of a Lebesgue summable function is itself summable, that is,

$$(16.1) \quad |f| \in L(v, R) \quad \text{for all } f \in L(v, R).$$

Thus the space $L(v, R)$ of Lebesgue summable functions forms a linear lattice.

The following theorem is due to P. Fatou [14] and is known in the literature as Fatou's Lemma.

Theorem 16.10 (Fatou's Lemma). Assume that (X, V, v) is a measure space over the prering V . Given a sequence $f_n \in L(v, R)$ consisting of nonnegative Lebesgue summable functions such that the sequence of integrals

$$\int f_n dv$$

is bounded, then the function $f = \liminf f_n$ belongs to the Lebesgue space $L(v, R)$ and we have the inequality

$$\int f dv = \int \liminf f_n dv \leq \liminf \int f_n dv.$$

Proof. For fixed n define the sequence

$$g_{nk} = f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k} \quad \text{for all } k = 0, 1, 2, \dots$$

Since the functions g_{nk} belong to the Lebesgue space $L(v, R)$ and are nonnegative and for fixed n form a decreasing sequence g_{nk} ($k = 1, 2, \dots$) with respect to relation less or equal \leq , from Monotone Convergence Theorem there exists a function $g_n \in L(v, R)$ such that

$$g_n = \lim_k g_{nk} \quad a.e. \quad \text{and} \quad \int g_n dv = \lim_k \int g_{nk} dv.$$

Notice that

$$g_n(x) = \inf \{f_k(x) : k \geq n\} \quad a.e. \quad \text{for all } n.$$

Since

$$(16.2) \quad g_n \leq g_{nk} \leq f_{n+k} \quad a.e. \quad \text{for all } n \text{ and } k$$

and the sequence of integrals $\int f_n dv$ is bounded, the sequence of integrals $\int g_n dv$ is bounded. Since the sequence g_n monotonically converges almost everywhere to the function

$$f = \liminf f_n \quad a.e.$$

by Monotone Convergence Theorem we get that $f \in L(v, R)$ and

$$\lim_n \int g_n dv = \int f dv.$$

On the other hand we have for fixed n from the estimate (16.2)

$$\int g_n dv \leq \liminf_k \int f_{n+k} dv = \liminf_k \int f_k dv.$$

Thus passing to the limit $n \rightarrow \infty$ in the above relation we get

$$\int f dv = \int \liminf_k f_k dv \leq \liminf_k \int f_k dv.$$

□

Theorem 16.11 (Lebesgue's Dominated Convergence Theorem). *Let (X, V, v) be a positive measure space on a prepring V and Y a Banach space.*

Assume that we are given a sequence $f_n \in L(v, Y)$ of Bochner summable functions that can be majorized by a Lebesgue summable function $g \in L(v, R)$, that is, for some null set $A \in N$ we have the estimate

$$|f_n(x)| \leq g(x) \quad \text{for all } x \notin A \text{ and } n = 1, 2, \dots$$

Then the condition

$$f_n(x) \rightarrow f(x) \quad a.e. \quad \text{on } X$$

implies the relations

$$f \in L(v, Y) \quad \text{and} \quad \|f_n - f\| \rightarrow 0$$

and, therefore, also the relations

$$\int f_n dv \rightarrow \int f dv \quad \text{and} \quad \int u(f_n, d\mu) \rightarrow \int u(f, d\mu)$$

for every bilinear continuous operator u from the product $Y \times Z$ into the Banach space W and any vector measure $\mu \in K(v, Z)$.

Proof. For proof of this theorem see Theorem 5 of Bogdanowicz [4, page 497]. \square

Theorem 16.12 (On absolutely summable series in $L(v, Y)$). *Let (X, V, v) be a positive measure space on a pre-ring V and Y a Banach space. Assume that we are given a sequence $f_n \in L(v, Y)$ of Bochner summable functions.*

If $\sum_n \|f_n\| < \infty$ then there exist a Bochner summable function $f \in L(v, Y)$ and a Lebesgue summable function $g \in L(v, R)$ and a null set $A \in N$ such that we have

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{and} \quad \sum_{n=1}^{\infty} |f_n(x)| \leq g(x) \quad \text{for all } x \notin A$$

and moreover when $k \rightarrow \infty$ we have the relations

$$\begin{aligned} \left\| \sum_{n \leq k} f_n - f \right\| &\rightarrow 0, \\ \int \sum_{n \leq k} f_n dv &\rightarrow \int f dv, \\ \int u(\sum_{n \leq k} f_n, d\mu) &\rightarrow \int u(f, d\mu), \end{aligned}$$

or equivalently

$$\sum_{n=1}^{\infty} f_n = f,$$

in the sense of convergence in the space $L(v, Y)$ of Bochner summable functions, and we have the commutativity of the operations

$$\begin{aligned} \sum_{n=1}^{\infty} \int f_n dv &= \int \sum_{n=1}^{\infty} f_n dv \\ \sum_{n=1}^{\infty} \int u(f_n, d\mu) &= \int \sum_{n=1}^{\infty} u(f_n, d\mu) = \int u(\sum_{n=1}^{\infty} f_n, d\mu), \end{aligned}$$

for every bilinear continuous operator u from the product $Y \times Z$ into the Banach space W and any vector measure $\mu \in K(v, Z)$.

17. A CHARACTERIZATION OF BOCHNER SUMMABLE FUNCTIONS

In this section we shall present several theorems that will be utilized in the following sections.

Proposition 17.1 (A characterization of Bochner summable functions). *Assume that (X, V, v) is a measure space on a pre-ring V of subsets of an abstract space X and Y a Banach space.*

Denote by G the set of non-negative Lebesgue summable functions $g \in L(v, R)$ being a limit of an increasing sequence of simple functions converging almost everywhere to the function g .

A function f mapping X into Y belongs to the space $L(v, Y)$ of Bochner summable functions, if and only if, there exist a sequence $s_n \in S(V, Y)$ of simple functions and a function $g \in G$, such that $s_n(x) \rightarrow f(x)$ almost everywhere on X and

$$|s_n(x)| \leq g(x) \quad \text{for all } n = 1, 2, \dots \quad \text{and almost all } x \in X.$$

Proof. If $f \in L(v, Y)$ then there exists a basic sequence of the form

$$s_n = h_1 + h_2 + \dots + h_n$$

converging almost everywhere to the function f . Notice that the sequence

$$S_n = |h_1| + |h_2| + \dots + |h_n|$$

is nondecreasing and is basic. Thus it converges almost everywhere to some summable function $g \in G$. Since

$$|s_n(x)| \leq S_n(x) \leq g(x) \quad \text{for almost all } x \in X,$$

we get the necessity of the condition.

The sufficiency of the condition follows from the Dominated Convergence Theorem. \square

Theorem 17.2 (Summability of $u(f, g)$). *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X . Let Y, Z, W be Banach spaces and u a bilinear bounded operator from the product $Y \times Z$ into W .*

Assume that $f \in L(v, Y)$ and $g \in L(v, Z)$ and either f or g is bounded almost everywhere on X .

Then the composed function $u(f, g) = h$ belongs to the space $L(v, W)$, where

$$h(x) = u(f(x), g(x)) \quad \text{for all } x \in X.$$

Proof. Assume that $f_n \in S(V, Y)$ is a basic sequence converging almost everywhere to the function f and $g_n \in S(V, Z)$ a basic sequence converging to g . Let \tilde{f} and \tilde{g} be the majorants of f and g , respectively, from the family G as in the preceding theorem.

Then the sequence $h_n = u(f_n, g_n)$ represents a sequence of simple functions from $S(V, W)$ and it can be majorized by the function $m(\tilde{f} + \tilde{g}) \in G$, where either

$$|f(x)| \leq m \quad \text{or} \quad |g(x)| \leq m \quad \text{for almost all } x \in X.$$

By continuity of the operator u we get

$$h_n(x) \rightarrow h(x) \quad \text{for almost all } x \in X.$$

Thus by Theorem 17.1 we get $h \in L(v, W)$. \square

Proposition 17.3 (Some product summability). *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space. Let*

$$u(y) = \frac{1}{|y|} y \text{ if } |y| > 0 \quad \text{and} \quad u(y) = 0 \text{ if } |y| = 0.$$

Assume that $f \in L(v, Y)$ and $g \in L(v, R)$. Then the product function $u \circ f \cdot g$ is summable that is $u \circ f \cdot g \in L(v, Y)$, where $u \circ f$ denotes the composition $(u \circ f)(x) = u(f(x))$ for all $x \in X$.

Proof. Take any natural number k and define a function

$$u_k(y) = (k|y| \wedge 1) \frac{1}{|y|} y \quad \text{for all } y \in Y, |y| > 0 \quad \text{and} \quad u_k(0) = 0.$$

Notice that the function u_k is continuous and

$$\lim_k u_k(y) = u(y) \quad \text{and} \quad |u_k(y)| \leq 1 \quad \text{for all } y \in Y.$$

Let s_n be a basic sequence converging almost everywhere to f and S_n a basic sequence converging almost everywhere to g . Let $S \in G$ be a majorant for the sequence S_n . Then we have that the sequence

$$h_{kn} = u_k \circ s_n \cdot S_n \in S(V, Y)$$

consists of simple functions and when $n \rightarrow \infty$ it converges almost everywhere to the function

$$h_k = u_k \circ f \cdot g.$$

Since S majorizes the sequence h_{kn} , from the Dominated Convergence Theorem we get $h_k \in L(v, Y)$ and moreover

$$|h_k(x)| \leq S(x) \quad \text{for almost all } x \in X.$$

Passing to the limit $k \rightarrow \infty$ and applying the Dominated Convergence Theorem yields that

$$u \circ f \cdot g \in L(v, Y).$$

□

18. SUMMABLE SETS

Assume now again that we have a measure space (X, V, v) on a prering V of subsets of an abstract space X . Following Bogdanowicz [6] and [7] denote by V_c the family of all sets $A \subset X$ whose characteristic function c_A is v -summable that is $c_A \in L(v, R)$. Put $v_c(A) = \int c_A dv$ for all sets $A \in V_c$.

Definition 18.1 (Summable sets and completion of a measure). *The family V_c will be called the family of **summable sets** and the set function v_c from V_c into R will be called the **completion** of the measure v .*

From Theorem 16.6 concerning properties of Lebesgue summable functions we can deduce the following proposition.

Proposition 18.2 (Summable sets form a delta ring). *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X .*

Then the family V_c of summable sets forms a δ -ring and v_c forms a measure. If in addition $X \in V_c$ then V_c forms a σ -algebra.

Proof. The space $L(v, R)$ forms a linear lattice. Thus from the identities

$$c_{A \cup B} = c_A \vee c_B \quad \text{and} \quad c_{A \cap B} = c_A \wedge c_B \quad \text{and} \quad c_{A \setminus B} = c_A - c_A \wedge c_B$$

we can conclude that the family V_c of summable sets forms a ring.

Now if $A_n \in V_c$ is a sequence of summable sets and $B_n = \bigcap_{j \leq n} A_j$ and $B = \bigcap_{j \geq 1} A_j$, from the Dominated Convergence Theorem 16.11 and from the relations

$$|c_{B_n}(x)| = c_{B_n}(x) \leq c_{A_1}(x) \quad \text{and} \quad c_{B_n}(x) \rightarrow c_B(x) \quad \text{for all } x \in X$$

we get that $B \in V_c$. Thus the family V_c of summable sets forms a δ -ring.

In the case, when $X \in V$, we get from the de Morgan law and the fact that V_c forms a δ -ring that

$$\bigcup_{n \geq 1} A_n = X \setminus \bigcap_{n \geq 1} (X \setminus A_n) \in V_c.$$

Hence in this case V_c forms a σ -algebra.

To show that the triple (X, V_c, v_c) forms a positive measure space assume that $A \in V_c$ and a sequence of disjoint sets $A_n \in V_c$ forms a decomposition of the set A . So $A = \bigcup_{j \geq 1} A_j$. Let $B_n = \bigcup_{j \leq n} A_j$. From the Dominated Convergence Theorem 16.11 and from the relations

$$|c_{B_n}(x)| = c_{B_n}(x) \leq c_A(x) \quad \text{and} \quad c_{B_n}(x) \rightarrow c_A(x) \quad \text{for all } x \in X$$

and linearity of the integral, we get that

$$\begin{aligned} v_c(A) &= \lim_n v_c(B_n) = \lim_n \int c_{B_n} dv \\ &= \lim_n \sum_{j \leq n} \int c_{A_j} dv = \lim_n \sum_{j \leq n} v_c(A_j) = \sum_j v_c(A_j). \end{aligned}$$

Thus v_c is countably additive on the delta ring V_c . \square

19. SUMMABILITY ON SETS

Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space.

A vector measure μ from a prering V into a Banach space Y is said to be of **finite variation** on V if

$$|\mu|(A) = \sup \left\{ \sum_{t \in T} |\mu(A_t)| : A = \bigcup_{t \in T} A_t \right\} < \infty \quad \text{for all } A \in V$$

where the supremum is taken over all finite disjoint decompositions $A_t \in V$ ($t \in T$) of the set $A = \bigcup_{t \in T} A_t$. The set function $|\mu|$ is called the **variation** of the vector measure μ .

We shall say that a function $f : X \rightarrow Y$ is **summable on a set** $A \subset X$ if the product function $c_A f$ belongs to the space $L(v, Y)$ of Bochner summable functions and we shall write

$$\int_A f dv = \int c_A f dv$$

to denote the integral of the function f on the set A .

Denote by V_f the family of all sets $A \subset X$ on which the function f is summable. Notice that the family V_c of summable sets can be thought of as family of sets on which the characteristic function c_X of the entire space X is summable.

Proposition 19.1 (Sets on which a function is summable form a δ -ring). *Assume that f is a function from the space X into the Banach space Y .*

Then the family of sets V_f forms a δ -ring and the set function

$$\mu(A) = \int_A f \, dv \quad \text{for all } A \in V_f$$

forms a σ -additive vector measure of finite variation on V_f .

Proof. Assume that $A, B \in V_f$. Then $c_A f \in L(v, Y)$ and $|c_B f| \in L(v, R)$. From Proposition 17.3 we get

$$c_{A \cap B} f = c_A c_B u \circ f \cdot |f| = u \circ (c_A f) \cdot |c_B f| \in L(v, Y).$$

Thus $A \cap B \in V_f$. It follow from linearity of the space $L(v, Y)$ and the identities

$$c_{A \setminus B} = c_A - c_{A \cap B} \quad \text{and} \quad c_{A \cup B} = c_{A \setminus B} + c_{A \cap B} + c_{B \setminus A}$$

that V_f forms a ring.

Now using the Dominated Convergence Theorem we can easily prove that V_f forms a δ -ring and the set function

$$\mu(A) = \int_A f \, dv \quad \text{for all } A \in V_f$$

forms a σ -additive vector measure and

$$|\mu|(A) \leq \int_A |f| \, dv < \infty \quad \text{for all } A \in V_f.$$

□

The family of sets V_f on which a function f is summable may consist only of the empty set. However in the case of a summable function this family is rich as follows from the following corollary.

Corollary 19.2 (Sets on which a summable function is summable form σ -algebra). *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y is a Banach space.*

If $f \in L(v, Y)$ is a summable function then the family V_f of sets, on which f is summable, forms a σ -algebra containing all summable sets, that is, we have the inclusion

$$V \subset V_c \subset V_f$$

and the set function $\mu(A) = \int_A f \, dv$ is σ -additive of finite total variation $|\mu|(X) \leq \int |f| \, dv$.

Proof. To prove this corollary notice that similarly as before we can prove that product gf of a summable bounded function $g \in L(v, R)$ with a summable function $f \in L(v, Y)$ is summable $gf \in L(v, Y)$. This implies that $V \subset V_c \subset V_f$. □

For further studies of vector measures we recommend Dunford and Schwartz [12], and for extensive survey of the state of the art in the theory of vector measures see the monograph of Diestel and Uhl [11]. Compare also Bogdanowicz and Oberle [9].

20. MEASURES GENERATING THE SAME INTEGRATION

In view of the existence of a variety of extensions of a measure from a prering onto δ -rings and the multiplicity of extensions to Lebesgue measures it is important to be able to identify measures that generate the same class of Lebesgue-Bochner summable functions $L(v, Y)$ and the same trilinear integral $\int u(f, d\mu)$ and thus the ordinary Bochner integral $\int f dv$. In this regard we have the following theorems.

Assume that (X, V_j, v_j) , $(j = 1, 2)$ are two measure spaces over the same abstract space X and Y, Z, W are any Banach spaces and U is the Banach space of bilinear bounded operators from the product $Y \times Z$ into W .

Theorem 20.1 (When $L(v_2, Y)$ extends $L(v_1, Y)$?). *For every Banach space Y we have $L(v_1, Y) \subset L(v_2, Y)$ and*

$$\int f dv_1 = \int f dv_2 \quad \text{for all } f \in L(v_1, Y)$$

if and only if $V_{1c} \subset V_{2c}$ and

$$v_{1c}(A) = v_{2c}(A) \quad \text{for all } A \in V_{1c}$$

that is the measure v_{2c} represents an extension of the measure v_{1c} .

Consequently we have the following theorem.

Theorem 20.2. *For any Banach space Y and any bilinear bounded transformation $u \in U$ we have $L(v_1, Y) = L(v_2, Y)$ and*

$$\int f dv_1 = \int f dv_2 \quad \text{for all } f \in L(v_1, Y)$$

and the spaces of vector measures $K(v_1, Z), K(v_2, Z), K(v_{1c}, Z), K(v_{2c}, Z)$ are isometric and isomorphic and

$$\int u(f, d\mu_1) = \int u(f, d\mu_2) = \int u(f, d\mu_{1c}) = \int u(f, d\mu_{2c}) \quad \text{for all } f \in L(v_1, Y),$$

where $\mu_1, \mu_2, \mu_{1c}, \mu_{2c}$ are vector measures that correspond to each other through the isomorphism, if and only if, the completions of the measures v_1, v_2 coincide $v_{1c} = v_{2c}$.

For proofs of the above theorems see Bogdanowicz [7]. It is important to relate the above theorems to the classical spaces of Lebesgue and Bochner summable functions and the integrals generated by Lebesgue measures.

Since there is a great variety of approaches to construct these spaces we shall understand by a classical construction of the Lebesgue space $L(\mu, R)$ the construction developed in Halmos [18] and by classical approach to the theory of the space $L(\mu, Y)$ of Bochner summable functions as presented in Dunford and Schwartz [12].

Now if (X, V, v) is a measure space on a prering V and (X, M, μ) represents a Lebesgue measure space where μ is the smallest extension of the measure v to a Lebesgue complete measure on the σ -ring M , then we have the following theorem.

Theorem 20.3. *For every Banach space Y the spaces $L(v, Y)$ and $L(\mu, Y)$ coincide and we have*

$$\int_A f dv = \int_A f d\mu \quad \text{for all } f \in L(\mu, Y) \quad \text{and } A \in M.$$

The above theorem is a consequence of the theorems developed in Bogdanowicz [5], page 267, Section 7.

21. CONTINUITY AND SUMMABILITY ON LOCALLY COMPACT SPACES

Now let the measure space (X, V, v) be the *Riemann measures space* over the Euclidean space $R^m = X$ on the pre-ring V of all cubes of the form

$$A = J_1 \times \dots \times J_m,$$

where J_i denotes an interval with end points $a_i \leq b_i$

or let X be a locally compact Hausdorff space, the family V consist of all sets $A = F \cap G$, where F is a compact set and G is an open set, and let v be any positive measure on V .

Theorem 21.1 (Continuous functions on compact sets are summable). *Assume that the triple (X, V, v) represents either the Riemann measure space or the measure space as defined above over a locally compact Hausdorff space X .*

If a function f is continuous from a compact set $Q \subset X$ into the Banach space Y , then the function is Bochner summable on the set Q that is $c_Q f \in L(v, Y)$ and thus the integrals

$$\int_Q f dv \quad \text{and} \quad \int_Q u(f, d\mu) = \int u(c_Q f, d\mu)$$

exist and moreover we have the estimates

$$\left\| \int_Q f dv \right\| \leq \int_Q \|f\| dv \quad \text{and} \quad \left\| \int_Q u(f, d\mu) \right\| \leq |u| \|\mu\| \int_Q \|f\| dv$$

for all $\mu \in K(v, Z)$, and $u \in U$.

Proof. Consider first the case of the Euclidean space R^m . In this case there exists a cube A such that $Q \subset A$. There exist disjoint cubes of diameter less than $1/n$, $A_i^n (i = 1, \dots, k_n)$, such that the intersections $A_i^n \cap Q$ are nonempty and $Q \subset \bigcup_i A_i^n \subset A$. Pick a point x_i^n from each of the sets $A_i^n \cap Q$.

In the case of a locally compact space consider the compact set $f(Q) = K \subset Y$. There exist nonempty disjoint sets $B_i^n (i = 1, \dots, k_n)$ of diameter less than $1/n$ such that $\bigcup_i B_i^n = K$ and each of the sets B_i^n is the intersection of an open set with a closed set. Therefore, $A_i^n = f^{-1}(B_i^n) \cap Q \in V$. Choose a point x_i^n in each of the sets A_i^n .

Define $f_n(x) = \sum_i f(x_i^n) c_{A_i^n}$. We have $f_n(x) \rightarrow c_Q(x)f(x)$ for every $x \in X$, and the sequence $|f_n(x)|$ is dominated either by $M c_A$ or by $M c_Q$. Notice that $f_n \in S(V, Y)$ by construction of the sequence. Using the Dominated Convergence Theorem 16.11 we conclude the proof. \square

In the case when the space X is topological, we say that the **measure is regular** if

$$v(A) = \inf\{v(E) : A \subset \text{int}(E), E \in V\} = \sup\{v(E) : \text{clo}(E) \subset A, E \in V\},$$

where $\text{int}(E)$ denotes the interior and $\text{clo}(E)$ the closure of the set E . The measure defined in the Euclidean space R^m by the formula

$$v(J_1 \times \dots \times J_m) = (b_1 - a_1) \cdots (b_m - a_m)$$

is regular.

Denote by $C(Y)$ the family of all functions f from the set X into the space Y such that there exists an increasing sequence of compact sets Q_n with the following property

$$\{x \in X : f(x) \neq 0\} \setminus \bigcup_n Q_n \in N,$$

the function f restricted to each of the sets Q_n is continuous on Q_n , and

$$\sup \int_{Q_n} |f| dv < \infty.$$

Such functions will be called **almost σ -continuous** (sigma continuous.)

The following theorem is equivalent to a theorem of Lusin [21]. One can prove it using the fact that continuous functions on compact sets are summable 21.1, and the Dominated Convergence Theorem 16.11, and Riesz-Egoroff property of a basic sequence 15.2.

Theorem 21.2 (Lusin's Theorem). *If (X, V, v) is the Riemann measure space over $X = R^n$, that is, the prering V consists of cubes as above, and the measure v represents the volume of the n -dimensional cube, then the space of Bochner summable functions coincides with the space of almost σ -continuous functions, that is, we have the identity*

$$C(Y) = L(v, Y).$$

To prove the following generalization of Lusin's theorem one may use in addition to previously mentioned properties the fact that in a locally compact topological Hausdorff space one can separate any two disjoint closed sets by means of a continuous function. This fact is known as Urysohn's Theorem, see Yosida [26], page 7.

For a complete proof of the following theorem see Bogdanowicz [8], page 230, Theorem 9, Part 3, and the definition of the family $C_\sigma(Y)$ on page 221, preceding Theorem 1.

Theorem 21.3 (Generalized Lusin's Theorem). *If X is a locally compact Hausdorff topological space and prering V consists of sets of the form $Q \cap G$, where Q is compact and G is an open set, then every almost σ -continuous function is Bochner summable $C(Y) \subset L(v, Y)$.*

If measure is regular, then the space of Bochner summable functions coincides with the space of almost σ -continuous functions

$$C(Y) = L(v, Y).$$

22. EXTENSIONS TO LEBESGUE MEASURES

If V is any nonempty collection of subsets of an abstract space X denote by V^σ the collection of sets that are countable unions of sets from V and denote by V^r the collection

$$V^r = \{A \subset X : A \cap B \in V \text{ for all } B \in V\}.$$

Now assume that (X, V, v) is a measure space and V_c the δ -ring of summable sets and $v_c(A) = \int c_A dv$. It is easy to prove that V_c^σ forms the smallest σ -ring containing the pre-ring V and the family N of v -null sets (see Bogdanowicz [5], section 3, page 258).

Moreover the set function defined by

$$(22.1) \quad \mu(A) = \sup \{v_c(B) : B \subset A, B \in V_c\} \text{ for all } A \in V_c^\sigma$$

forms a Lebesgue measure on V_c^σ .

A Lebesgue measure is called **complete** if all subsets of sets of measure zero are in the domain of the measure and thus have measure zero. The above Lebesgue measure μ can be characterized as the smallest extension of the measure v to a complete Lebesgue measure. This fact follows from Part 7, of Theorem 4, of [5], page 259. Hence this measure is unique.

For any fixed function f from X into a Banach space Y denote by V_f the collection of all sets $A \subset X$ such that $c_A f \in L(v, Y)$. The family V_f may consist of just the empty set but in the case of a summable function $f \in L(v, Y)$ it represents a sigma algebra extending the family V_c of summable sets.

One can prove the following identity

$$V_c^r = \bigcap_{f \in L(v, Y)} V_f = \bigcap_{f \in L(v, R)} V_f.$$

The family V_c^r as an intersection of σ -algebras forms itself a σ -algebra containing the δ -ring V_c . The smallest σ -algebra V^a containing V_c is given by the formula

$$V^a = \{A \subset X : A \in V_c^\sigma \text{ or } X \setminus A \in V_c^\sigma\}.$$

If $X \in V_c^\sigma$ then the sigma algebras coincide $V_c^\sigma = V^a = V_c^r$. If $X \notin V_c^\sigma$, one can always extend the measure v to a Lebesgue measure on V^a or V_c^r by the formula

$$(22.2) \quad \mu(A) = v_c(A) \text{ if } A \in V_c \text{ and } \mu(A) = \infty \text{ if } A \notin V_c.$$

However if $\sup \{v_c(A) : A \in V_c\} = a < \infty$ and $X \notin V_c^\sigma$ the extensions are not unique. Indeed one can take in this case $\mu(X) = b$, where b is any number from the interval $[a, \infty)$, and put

$$\mu(A) = v_c(A) \text{ if } A \in V_c \text{ and } \mu(A) = b - v_c(X \setminus A) \text{ if } X \setminus A \in V_c$$

to extend the measure v_c onto the σ -algebra V^a preserving sigma additivity.

Consider an example. Let (X, V, v) be the following measure space:

$$X = R, \quad V = \{\emptyset, \{n\} : n = 1, 2, \dots\}, \quad v(\emptyset) = 0, \quad v(\{n\}) = 2^{-n}.$$

In this case the family N of null sets contains only the empty set \emptyset , the family S of simple sets consists of finite subsets of the set of natural numbers \mathcal{N} , the family V_c of summable sets consists of all subsets of \mathcal{N} , we have $V_c^\sigma = V_c$, the smallest σ -algebra extending V_c consists of sets that either are subsets of \mathcal{N} or their complements are

subsets of \mathcal{N} , finally $V_c^r = P(R)$ consists of all subsets of R . Since $v_c(\mathcal{N}) = 1$ is the supremum of v_c , the measure v_c has many extensions onto the σ -algebras V^a and V_c^σ . An infinite valued extension onto the power set $P(R)$ is given by the formula (22.2) and a totally finite valued extension, for instance, by

$$\mu(A) = \sum_{n \in A \cap \mathcal{N}} 2^{-n} \quad \text{for all } A \subset R.$$

23. EXTENSION TO OUTER MEASURE

In the proofs of some theorems it is useful to use the notion of the outer measure also called the exterior measure in the literature.

We shall say that a nonempty family V of subsets of a space X is **hereditary** if it satisfies the implication

$$A \subset B \in V \implies A \in V.$$

Definition 23.1 (Outer measure). *A set function $\eta : V \mapsto [0, \infty]$ is called an **outer measure** if its domain V forms a hereditary sigma ring of subsets of an abstract space X and it takes value zero on the empty set, is monotone, and countably subadditive, that is it has the following properties*

- (1) $\eta(\emptyset) = 0$,
- (2) if $A \subset B \in V$ then $\eta(A) \leq \eta(B)$,
- (3) for any sequence $A_n \in V$ we have $\eta(\bigcup_n A_n) \leq \sum_n \eta(A_n)$.

Definition 23.2 (Set function v^*). *Assume that (X, V, v) is a positive measure space over the prering V .*

For any subset A of the space X define the set function v^ as follows if there exists a summable set B containing the set A*

$$v^*(A) = \inf \{v_c(B) : A \subset B \in V_c\}$$

else let $v^(A) = \infty$.*

Theorem 23.3 (Function v^* forms an outer measure extending v_c). *Assume that (X, V, v) forms a positive measure space over the prering V and v^* represents the set function as defined above.*

Then

- (1) *if for some set $v^*(A) < \infty$ then there exists a summable set $B \in V_c$ such that $A \subset B$ and $v^*(A) = v_c(B)$,*
- (2) *the set function v^* forms an outer measure,*
- (3) *it extends the measure v_c from the family V_c of the summable sets onto the family of all subsets of the space X , and therefore it extends also the measure v .*

Proof. To prove part (1) take any positive integer n . Since the number

$$v^*(A) + 1/n$$

is greater than the greatest lower bound $v^*(A)$ of the set of numbers

$$D = \{v_c(B) : A \subset B \in V_c\},$$

the number $v^*(A) + 1/n$ is not a lower bound of the number set D . Therefore, for each n , there exists a summable set $B_n \in V_c$, such that

$$A \subset B_n \quad \text{and} \quad v^*(A) \leq v_c(B_n) < v^*(A) + 1/n.$$

Put $B = \bigcap_n B_n$. We have $A \subset B \in V_c$ and from monotonicity of the measure v_c we get

$$v^*(A) \leq v_c(B) \leq v_c(B_n) < v^*(A) + 1/n \quad \text{for all } n = 1, 2, \dots$$

Hence $v^*(A) = v_c(B)$.

As a consequence if $A \in V_c$ then $v^*(A) = v_c(A)$ thus the set function v^* extends the measure v_c from the family V_c of summable sets to all subsets of the space X .

Notice that since the empty set \emptyset is in V_c we must have $v^*(\emptyset) = v_c(\emptyset) = 0$.

To prove the monotonicity of the set function v^* , that is, the validity of the implication

$$A \subset B \implies v^*(A) \leq v^*(B),$$

notice that the above implication is always true when $v^*(B) = \infty$. To establish the validity of the implication when $v^*(B) < \infty$ apply part (1) of the theorem.

Finally to prove countable subadditivity

$$v^*\left(\bigcup_n A_n\right) \leq \sum_n v^*(A_n),$$

notice that the inequality is always true when the right side is infinite. So consider the case when it is finite. In this case each term of the sum must be finite and so there exist summable sets $B_n \in V_c$ such that

$$A_n \subset B_n \quad \text{and} \quad v^*(A_n) = v_c(B_n) \quad \text{for all } n = 1, 2, \dots$$

Thus we have

$$A = \bigcup_n A_n \subset \bigcup_n B_n = B.$$

Consider the set B . Introduce the sets

$$D_1 = B_1, \quad D_n = B_n \setminus (B_1 \cup \dots \cup B_{n-1}) \quad \text{for all } n > 1$$

Notice that the sets $D_n \in V_c$ and they are disjoint. Moreover $D_n \subset B_n$ and for all m we have the estimate

$$\sum_{n \leq m} v_c(D_n) \leq \sum_{n \leq m} v_c(B_n) \leq \sum_{n=1}^{\infty} v_c(B_n) < \infty.$$

Since $B = \bigcup_n D_n$ the set B is in the ring V_c and from countable additivity of the measure v_c we have

$$v^*(A) \leq v_c(B) = \sum_n v_c(D_n) \leq \sum_n v_c(B_n) = \sum_n v^*(A_n).$$

The above inequality completes the proof. \square

The outer measure v^* has other convenient representations and in the case of topological spaces it can be defined by or related to topology. To do that we will need some theorems on approximation of sets.

We shall use the following notation. If V is any nonempty family of sets then V^σ will denote the family of all sets that are countable unions of sets from the family V , and by V^δ the family of all sets that are countable intersections of sets from the family V . We shall use the following shortcut notation

$$V^{\sigma\delta} = (V^\sigma)^\delta.$$

We remind the reader that if V is any collection of sets containing at least the empty set \emptyset then $S(V)$ denotes the collection of simple sets consisting of finite disjoint unions of sets from the collection V .

The operation \div will denote the symmetric difference of sets

$$A \div B = (A \setminus B) \cup (B \setminus A).$$

The collection of all subsets of a space X with this operation forms an Abelian group.

The following theorem is a consequence of Theorem 4, Part 6, page 259, of Bogdanowicz [5]. We shall present its proof for the sake of completeness, since it is essential in the sequel development of the theory.

Theorem 23.4 (A characterization of summable sets). *Assume that (X, V, v) is a positive measure space. Assume that the pre-ring V contains the space X . Let $S = S(V)$ and N denote the collection of null sets.*

Then a set $A \subset X$ is summable, that is $A \in V_c$, if and only if one of the following conditions is satisfied

- (1) *There is a set $B \in S^{\delta\sigma}$ and $D \in N$ such that $A = B \div D$.*
- (2) *There is a set $B \in V^{\sigma\delta}$ and $D \in N$ such that $A = B \div D$.*

Proof. Assume that the set A is summable. Denote by S the collection $S(V)$ of simple sets. It follows from the definition of a summable function that there exists a sequence of simple functions $s_n \in S(V, R)$ and a null set $D \in N$ such that $s_n(x) \rightarrow c_A(x)$ if $x \notin D$. Put

$$f_n(x) = \inf \{s_m(x) : m > n\}.$$

The sequence of values $f_n(x)$ increasingly converges to the value $c_A(x)$ for every point $x \notin D$.

Denote by B the set of all points $x \in X$ for which there exists an index n such that $f_n(x) > 0$. We see that $B = \bigcup_{np} B_{np}$ where $B_{np} = \{x \in X : f_n(x) \geq 1/p\}$ and further $B_{np} = \bigcap_{m \geq p} E_{mp}$ where $E_{mp} = \{x \in X : s_m(x) \geq 1/p\}$. Since $E_{mp} \in S$ therefore $B \in S^{\delta\sigma}$.

We notice that $A \div B \subset D$. Hence the set $C = A \div B$ as a subset of the null set D is itself a null set. From the properties of the symmetric difference we get $A = (B \div B) \div A = B \div (B \div A) = B \div C$. Thus the necessity of the condition is proved.

To prove the sufficiency of the condition, take any set $B \in S^{\delta\sigma}$. The set B obviously belongs to the sigma algebra V_c . Let $A = B \div C$ where $C \in N$. The characteristic functions c_A and c_B are equal almost everywhere.

Since $c_B \in V_c$ therefore there exists a basic sequence of simple functions $s_n \in S(V, R)$ converging almost everywhere to the function c_B and therefore also converging almost everywhere to the function c_A . Thus we have $c_A \in L(v, R)$, that is, $A \in V_c$. Thus the condition (1) is equivalent to summability of the set A .

Now assume that the condition (1) is satisfied. Thus

$$A = B \div D, \quad B \in S^{\delta\sigma}, \quad D \in N.$$

Since V_c forms a sigma algebra we have $A_1 = X \div B \in V_c$. Thus A_1 must satisfy the condition (1). Therefore there exists sets B_1 and D_1 such that

$$A_1 = B_1 \div D_1, \quad B_1 \in S^{\delta\sigma}, \quad D_1 \in N.$$

We have

$$\begin{aligned} A &= X \div X \div B \div D \\ &= X \div (X \div B) \div D \\ &= X \div A_1 \div D \\ &= X \div B_1 \div D_1 \div D. \end{aligned}$$

Since $X \div B_1 = X \setminus B_1$, it follows from De Morgan law on complements of sets that the set B_2 satisfies the relations

$$B_2 = X \setminus B_1 \in S^{\sigma\delta} = V^{\sigma\delta}.$$

Since the set $D_2 = D_1 \div D \subset D_1 \cup D$ as a subset of a null set it is itself a null set. Thus we have the required representation $A = B_2 \div D_2$. Thus from condition (1) we derived the condition (2). It is easy to see from the symmetry of the argument that assuming that condition (2) holds for any summable set then the condition (1) must be also true. \square

Proposition 23.5 (For a prering V family V^σ is closed under finite intersection).
Let X be an abstract space and V a prering of subsets of X . If $A, B \in V^\sigma$ then $A \cap B \in V^\sigma$.

Proof. Assume $A = \cup_m A_m$ and $B = \cup_n B_n$ where the sequences A_m and B_n are from the prering V . Then

$$A \cap B = \bigcup_{m,n} (A_m \cap B_n).$$

By definition of a prering each set $A_m \cap B_n$ is a union of a finite number of sets from the family V . Thus $A \cap B \in V^\sigma$. \square

Proposition 23.6. Assume that (X, V, v) is a positive measure space. Assume that the prering V forms a pre-algebra, that is $X \in V$.

Then for every summable $A \in V_c$ and every $\varepsilon > 0$ there exists a set $B \in V^\sigma$ such that

$$A \subset B \quad \text{and} \quad v_c(B) \leq v_c(A) + \varepsilon.$$

Proof. Take any summable set A and any $\varepsilon > 0$. By Theorem 23.4 there exists a sequence $B_n \in V^\sigma$ such that $A = B \div D$ where $B = \bigcap_n B_n$ and D is null set.

By Proposition 23.5 the family of sets V^σ is closed under finite intersections. Thus we have

$$B'_n = \bigcap_{j \leq n} B_j \in V^\sigma \quad \text{for all } n.$$

The sequence of characteristic functions $c_{B'_n}$ of sets B'_n decreasingly converges to the characteristic function c_B and therefore it converges almost everywhere to the characteristic function c_A . The sequence is bounded by summable function, namely by the characteristic function c_X . By Lebesgue's Dominated Convergence Theorem

$$v_c(B'_n) \rightarrow v_c(A).$$

Since D is a null set, there exists a sequence $C_j \in V$ such that

$$\sum_{j=1}^{\infty} v(C_j) < \infty \quad \text{and} \quad D \subset D_n = \bigcup_{j>n} C_j \quad \text{for all } n.$$

The sequence D_n is decreasing and since

$$v_c(D) \leq \sum_{j>n} v(C_j) \rightarrow 0.$$

Thus we have

$$A \subset B'_n \cup D_n \in V^\sigma \quad \text{for all } n,$$

and

$$v_c(A) \leq v_c(B'_n \cup D_n) \leq v_c(B'_n) + v_c(D_n) \quad \text{for all } n.$$

Thus for sufficiently large index n we will have

$$v_c(A) \leq v_c(B'_n \cup D_n) \leq v_c(A) + \varepsilon.$$

□

Proposition 23.7. *Let X be an abstract space and V a prering of subsets of X . If $A \in V^\sigma$ then there exists a sequence I_n of disjoint sets from the prering V such that*

$$A = \bigcup_n I_n.$$

Proof. By definition of the family V^σ there exists a sequence $A_n \in V$ such that $A = \bigcup_n A_n$. Consider the sequence defined recursively by

$$B_1 = A_1, \quad B_n = A_n \setminus \bigcup_{j<n} A_j \quad \text{for all } n > 1.$$

Since the family $S(V)$ of simple sets generated by the prering V forms a ring, each of the sets B_n is a union of a finite number of disjoint sets from V . Since the sequence B_n consists of disjoint sets, combining all the sets from the decompositions of sets B_n into a sequence I_m we get

$$A = \bigcup_n B_n = \bigcup_m I_m.$$

□

The following theorem represents a generalization of Proposition 23.6 to the case of arbitrary positive measure space over a prering.

Theorem 23.8 (Approximation by summable V^σ sets). *Assume that (X, V, v) is a positive measure space.*

Then for every summable $B \in V_c$ and every $\varepsilon > 0$ there exists a set $C \in V^\sigma$ such that

$$B \subset C \quad \text{and} \quad v_c(C) \leq v_c(B) + \varepsilon.$$

Proof. It follows from the definition of the space $L(v, R)$ of Lebesgue summable functions that the support of the characteristic function of the set B can be covered by a countable number of the sets from the prepring V .

Indeed assume that s_n is a basic sequence converging to the function c_B at every point $x \notin D$ where D is a null set. By definition of a null set D can be covered by a countable family F_1 of sets from the prepring V and the support of each simple function s_n consists of finite number of sets from V thus there exists a countable family F_2 of set from V covering the supports of all the functions s_n .

Denote by G the set covered by the family $F_1 \cup F_2$ of sets. Plainly $G \in V^\sigma$ and at every point $x \notin G$ all functions $s_n(x)$ have value zero, so their limit must be also 0 and this means that $c_B(x) = 0$, and this means that is $x \notin B$. Therefore we must have the inclusion $B \subset G$.

By Proposition 23.7, there exists a sequence of disjoint sets

$$X_m \in V$$

such that $G = \bigcup_m X_m$. Let

$$V_m = \{I \in V : I \subset X_m\} \quad \text{and} \quad v_m(I) = v(I) \quad \text{for all} \quad I \in V_m.$$

Each triple (X_m, V_m, v_m) forms a measure space and, it is easy to see that, any function f whose support is in the set X_m belongs to the Lebesgue space $L(v_m, R)$ if and only if it belongs to the space $L(v, R)$. Thus for each m the set

$$B_m = X_m \cap B$$

is summable with respect to the measure space (X_m, V_m, v_m) .

Now take any $\varepsilon > 0$. Since $X_m \in V_m$ by Proposition 23.6 there exist sets

$$C_m \in V_m^\sigma \subset V^\sigma$$

such that

$$B_m \subset C_m \quad \text{and} \quad v_c(C_m) \leq v_c(B_m) + 2^{-m}\varepsilon \quad \text{for all} \quad m.$$

Put $C = \bigcup_m C_m$. Clearly $C \in V^\sigma$ and

$$B \subset C \quad \text{and} \quad v_c(C) \leq v_c(B) + \varepsilon,$$

which completes the proof. \square

Theorem 23.9 (Generating outer measure by V^σ sets). *Assume that (X, V, v) is a positive measure space over the prepring V .*

For any set $A \subset X$ let $\eta(A)$ be defined by the following formula if the set of sums

$$H = \inf \left\{ \sum_{n=1}^{\infty} v(B_n) < \infty : B_n \in V, A \subset \bigcup_n B_n \right\}$$

is nonempty then $\eta(A) = \inf H$ else let $\eta(A) = \infty$.

Then the set function η coincides with the outer measure v^ on all subsets A of the space X .*

Proof. We have the following inequality

$$v^*(A) \leq \eta(A) \quad \text{for all } A \subset X.$$

Indeed when $\eta(A) = \infty$ the inequality clearly is satisfied. When $\eta(A) < \infty$ consider any sequence $B_n \in V$ covering the set A and such that

$$\sum_{n=1}^{\infty} v(B_n) < \infty.$$

Since $B_n \in V \subset V_c$ by the Monotone Convergence Theorem we get that the set

$$B = \bigcup_n B_n$$

is summable $B \in V_c$. Since $A \subset B$ thus by definition of the set function v^* and by additivity and monotonicity of the measure v_c we have

$$v^*(A) \leq v_c(B) \leq \sum_{n=1}^{\infty} v_c(B_n) = \sum_{n=1}^{\infty} v(B_n).$$

Thus

$$v^*(A) \leq \eta(A) \quad \text{for all } A \subset X.$$

Now let us establish the opposite inequality

$$\eta(A) \leq v^*(A) \quad \text{for all } A \subset X.$$

This inequality is true when $v^*(A) = \infty$, so consider the case $v^*(A) < \infty$. In this case there exists a summable set $B \in V_c$ such that

$$A \subset B \quad \text{and} \quad v^*(A) = v_c(B).$$

Take any $\varepsilon > 0$. By Theorem 23.8 there exists a set $C \in V^\sigma$ such that

$$B \subset C \quad \text{and} \quad v_c(C) \leq v_c(B) + \varepsilon.$$

By Proposition 23.7 there exists a sequence of disjoint sets $I_n \in V$ such that $C = \bigcup_n I_n$. Since $A \subset B \subset C$ we get by definition of the set function η that

$$\eta(A) \leq \sum_n v(I_n) = v_c(C) \leq v_c(B) + \varepsilon = v^*(A) + \varepsilon.$$

Since ε was fixed but arbitrary the above relation implies

$$\eta(A) \leq v^*(A) \quad \text{for all } A \subset X.$$

Hence we have established that for any measure space (X, V, v) the following identity is true $v^*(A) = \eta(A)$ for all subsets A of the space X . \square

Definition 23.10 (Outer regularity). *Assume that X is a topological space and (X, V, v) a measure space. If for every set A in the prering V we have*

$$v(A) = \inf \{v(B) : A \subset \text{int}(B)\},$$

*where $\text{int}(B)$ denotes the interior of the set B , then such a measure space is called **outer regular**.*

Observe that the Riemann measure space (R, V, v) defined over the prering V of all bounded intervals of R is outer regular. The same is true for the Riemann measure space (R^n, V, v) where the prering V consists of all n -dimensional cubes.

Theorem 23.11. *Assume that X is a topological space and (X, V, v) is an outer regular measure space over the prepring V .*

Define a set function λ as follows. For any set $A \subset X$ if the set of sums

$$H = \left\{ \sum_{n=1}^{\infty} v(B_n) < \infty : B_n \in V, A \subset \bigcup_n \text{int}(B_n) \right\}$$

is nonempty then $\lambda(A) = \inf H$ else let $\lambda(A) = \infty$.

Then the set function λ coincides with the outer measure v^ on all subsets A of the space X .*

Proof. It is obvious that

$$v^*(A) \leq \lambda(A) \quad \text{for all } A \subset X.$$

To prove the opposite inequality take any $\varepsilon > 0$ and any sequence of sets $A_n \in V$ such that

$$\sum_n v(A_n) < \infty \quad \text{and} \quad A \subset \bigcup_n A_n.$$

By outer regularity of the measure space there exist sets $B_n \in V$ such that

$$A_n \subset \text{int}(B_n) \quad \text{and} \quad v(B_n) < v(A_n) + 2^{-n}\varepsilon \quad \text{for all } n.$$

We have

$$A \subset \bigcup_n \text{int}(B_n) \quad \text{and} \quad \sum_n v(B_n) \leq \sum_n v(A_n) + \varepsilon < \infty.$$

Thus

$$\lambda(A) \leq \sum_n v(B_n) \leq \sum_n v(A_n) + \varepsilon$$

that is

$$\lambda(A) - \varepsilon \leq \sum_n v(A_n).$$

Since $A_n \in V$ was an arbitrary sequence covering the set A and having finite sum $\sum_n v(A_n)$, the above inequality yields

$$\lambda(A) - \varepsilon \leq v^*(A).$$

Passing to the limit $\varepsilon \rightarrow 0$ in the above we get

$$\lambda(A) \leq v^*(A) \quad \text{for all } A \subset X.$$

Hence we have $\lambda(A) = v^*(A)$ for all $A \subset X$. □

Theorem 23.12 (Outer regularity of outer measure). *Assume that X is a topological space and (X, V, v) is an outer regular measure space over the prepring V .*

Then for every $\varepsilon > 0$ and every set $A \subset X$ such that $v^(A) < \infty$ there exists an open set G such that*

$$A \subset G \quad \text{and} \quad v^*(G) < v^*(A) + \varepsilon.$$

Proof. The proof follows from the monotonicity and subadditivity of the outer measure v^* and from the preceding theorem. □

24. PROPERTIES OF VITALI'S COVERINGS

In this section X will denote a close bounded interval $[a, b]$. Let (X, V, v) denote the Riemann measure space over the prering V of all subintervals I of the interval X and let $v(I)$ denote the length of the interval I . Plainly since $X \in V$ we have $v(I) \leq v(X) < \infty$ for all $I \in V$.

Definition 24.1 (Vitali covering). *We shall say that a family $H \subset V$ forms a **Vitali covering** of a set $E \subset X$ if H consists of closed sets including the empty set \emptyset and for every point $x \in E$ and every $\varepsilon > 0$ the family H contains a set I such that $x \in I$ and $0 < v(I) < \varepsilon$.*

Theorem 24.2 (Property of Vitali's covering). *Assume that the family H forms a Vitali covering of a set $E \subset X$. If H does not contain a finite disjoint family of closed sets I_j covering the set E then one can find a sequence of disjoint closed sets $I_j \in H$ such that*

- (1) *for each j the intersection $E \cap I_j$ is nonempty,*
- (2) *for each n the family H_n of sets defined by*

$$H_n = \{I \in H : E \cap I \neq \emptyset, 0 < v(I), I \cap I_j = \emptyset \forall j \leq n\}$$

is nonempty,

- (3) *for each n we have $I_{n+1} \in H_n$ and $\varepsilon_n < 2v(I_{n+1})$, where*

$$\varepsilon_n = \sup \{v(I) : I \in H_n\}.$$

Proof. Take any point $x \in E$. Let I_1 denote any interval from H containing the point x . Since I_1 by assumption cannot cover E the set $E \setminus I_1$ is nonempty. Thus there exists a point x_1 in E that does not belong to I_1 that is x_1 belongs to the set $X \setminus I_1$. Since complement of a closed set is open there is an open interval J containing x_1 such that $J \subset X \setminus I_1$. By definition of Vitali's covering there is a closed set $I \in H$ of sufficiently small length such that $x_1 \in I$ and $v(I) > 0$ and $I \subset J$. Hence the family of sets

$$H_1 = \{I \in H : E \cap I \neq \emptyset, 0 < v(I), I \cap I_1 = \emptyset\}$$

is nonempty. Thus the set $R_1 = \{v(I) : I \in H_1\}$ is nonempty and bounded from above by $v(X)$. Thus by axiom of completeness of the space R of reals the least upper bound ε_1 of the set R_1 is well defined and we have

$$0 < \varepsilon_1 = \sup \{v(I) : I \in H_1\}.$$

The number $\frac{1}{2}\varepsilon_1$ is smaller than the least upper bound ε_1 of the set R_1 . Therefore there exists a set $I_2 \in H_1$ such that $\frac{1}{2}\varepsilon_1 < v(I_2)$, that is

$$\varepsilon_1 < 2v(I_2).$$

Clearly the pair I_1, I_2 forms disjoint sets and the conditions (2) and (3) of the theorem are satisfied for $n = 1$.

It is also clear that we can continue by induction to obtain a sequence $I_j \in H$ of disjoint sets, and a sequence H_n of subfamilies of H , and a sequence ε_n of positive numbers, satisfying the conditions (1), (2), and (3) as required in the theorem. \square

Definition 24.3 (Sparse sequence). *Assume that H forms a Vitali covering of a set $E \subset X$. Any sequence of sets $I_j \in H$ satisfying the conditions (1), (2), and (3), of Theorem 24.2 will be called a **sparse sequence** with respect to the set E .*

Proposition 24.4 (For sparse sequence $\lim_n \varepsilon_n = 0$). *If $I_j \in H$ is a sparse sequence relatively to the set E and ε_j is the associated sequence satisfying the condition (3) of Theorem 24.2, then $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. Since $X \in V$ the family V_c of summable sets forms a sigma algebra. Thus the set $A = \bigcup_n I_n$ belongs to V_c . Since the sets I_n are disjoint, we have

$$\sum_n v(I_n) = \sum_n v_c(I_n) = v_c(A) \leq v_c(X) = v(X) < \infty.$$

From the convergence of the above series follows that $v(I_n) \rightarrow 0$. Since from condition (3) of Theorem 24.2 follows that $\varepsilon_n < 2v(I_{n+1})$ for all n we must have $\varepsilon_n \rightarrow 0$. \square

Definition 24.5 (Set operator S). *For any interval $I \in V$ define the set operator $S : V \mapsto V$ as follows: If I is the empty set \emptyset , let $S(I) = \emptyset$ else let*

$$S(I) = [c - 2v(I), d + 2v(I)] \cap X$$

where $c \leq d$ are the end points of the interval I .

Notice that we have a convenient estimate $v(S(I)) \leq 5v(I)$ for every $I \in V$.

Lemma 24.6. *Given an interval $I \in V$ and two points $x, y \in X$. If*

$$|x - y| \leq 2v(I) \quad \text{and} \quad y \in I$$

then the point x belongs to the interval $S(I)$.

Proof. The proof is obvious. \square

Definition 24.7 (Interlaced cover). *We shall say that the sequence $I_j \in V$ forms an **interlaced cover** of a set $E \subset X$ if the sets I_j are closed and disjoint and*

$$E \subset \left(\bigcup_{j \leq n} I_j \right) \cup \left(\bigcup_{j > n} S(I_j) \right) \quad \text{for all } n = 1, 2, \dots$$

Theorem 24.8 (Vitali Covering Theorem). *Assume that $E \subset X$ and H forms a Vitali covering of the set E . Then there exists a sequence $I_j \in H$ of intervals forming an interlaced covering of the set E .*

Proof. If the set E can be covered by a finite number of disjoint sets I_1, I_2, \dots, I_n from the family H , by setting $I_j = \emptyset$ for $j > n$ we get an interlaced cover from the family H of the set E .

If a finite cover from the family H does not exist then there exists a sparse sequence $I_j \in H$ relatively to the set E . Let the sequence of families $H_n \subset H$ and the sequence ε_n be defined as in Theorem 24.2. Since the sequence H_n of families is decreasing the sequence ε_n is non-increasing.

We shall prove that the sequence of sets I_j forms an interlaced cover of the set E . To this end take any index n and consider the set

$$B_n = E \setminus \bigcup_{j \leq n} I_j.$$

Since the set B_n is nonempty there exist a point $x \in E$ and an interval $I \in H$ such that $x \in I$ and $0 < v(I)$ and $I \cap I_j = \emptyset$ for all $j \leq n$. Thus $I \in H_n$ and so $v(I) \leq \varepsilon_n$. By Proposition 24.4 we must have $\varepsilon_m \rightarrow 0$. Thus for all sufficiently large indexes m we must have

$$\varepsilon_m < v(I).$$

Thus the set of indexes

$$K = \{m : v(I) \leq \varepsilon_m\}$$

contains the index n and is at most finite. Let k be the largest index belonging to K . Thus we must have $k \geq n$ and

$$\varepsilon_{k+1} < v(I) \leq \varepsilon_k.$$

The relation $v(I) \leq \varepsilon_k$ implies $I \in H_k$ and thus $I \cap I_j = \emptyset$ for all $j \leq k$. The relation $\varepsilon_{k+1} < v(I)$ implies that it is not true that $I \cap I_j = \emptyset$ for all $j \leq (k+1)$. Hence we must have $I \cap I_{k+1} \neq \emptyset$. So there exists a point $y \in X$ such that $y \in I$ and $y \in I_{k+1}$. Thus since $x \in I$ we have

$$|x - y| \leq v(I) \leq \varepsilon_k \leq 2v(I_{k+1}) \quad \text{and} \quad y \in I_{k+1}.$$

By Lemma 24.6 we must have $x \in S(I_{k+1})$, which implies the inclusion

$$E \subset \left(\bigcup_{j \leq n} I_j \right) \cup \left(\bigcup_{j > n} S(I_j) \right).$$

Since the index n was fixed but arbitrary the theorem is established. \square

25. LEBESGUE THEOREM ON DIFFERENTIABILITY OF MONOTONE FUNCTIONS

In this section we will prove Lebesgue's theorem asserting that every real-valued monotone function defined on an interval I has a finite derivative almost everywhere on I . To this end it is convenient to introduce derivatives known in the literature as Dini's derivatives.

25.1. Dini's derivatives. Consider a real-valued function f defined on an open interval (a, b) .

Definition 25.1. By a *right-sided upper Dini derivative* of f at a point $x \in (a, b)$ we shall understand the finite or infinite limit

$$D_r^+ f(x) = \limsup_{h>0, h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)).$$

Similarly we define the *right-sided lower Dini derivative* by

$$D_r^- f(x) = \liminf_{h>0, h \rightarrow 0} \frac{1}{h} (f(x+h) - f(x)),$$

and *left-sided upper derivative*

$$D_l^+ f(x) = \limsup_{h>0, h \rightarrow 0} \frac{1}{h} (f(x) - f(x-h)),$$

and *left-sided lower derivative*

$$D_l^- f(x) = \liminf_{h>0, h \rightarrow 0} \frac{1}{h} (f(x) - f(x-h)).$$

The function f has derivative $f'(x)$ at a point x if and only if all four Dini's derivatives are finite and have equal values. We have the following relations between Dini's derivatives.

$$D_r^- f(x) \leq D_r^+ f(x) \quad \text{and} \quad D_l^- f(x) \leq D_l^+ f(x) \quad \text{for all } x \in (a, b).$$

By an **increasing** function f on an interval I we shall understand a function satisfying the following condition

$$f(x) \leq f(y) \quad \text{for all } x < y, x, y \in I.$$

Such a function is also called in the literature nondecreasing.

By an **decreasing** function f on an interval I we shall understand a function satisfying the condition

$$f(x) \geq f(y) \quad \text{for all } x < y, x, y \in I.$$

A function that is either increasing or decreasing on an interval is called **monotone** on that interval.

25.2. Differentiability of monotone functions.

Theorem 25.2 (Lebesgue). *Let X denote a closed bounded interval $[a, b]$ and assume that (X, V, v) denote the Riemann measure space on the preimage V of all subintervals of X .*

If f is a monotone real-valued function on X , then the derivative $f'(x)$ exists in the open interval (a, b) and is finite for almost all x .

Proof. We may assume without loss of generality that the function f is increasing otherwise we would consider the function $g(x) = -f(x)$ for all $x \in I$.

Introduce a set function $w : V \mapsto R$ by the formula

$$w(I) = f(\beta) - f(\alpha),$$

where $\alpha \leq \beta$ are the end points of the interval I . The set function w is finitely additive on the prering V and as such it admits a unique extension to a finitely additive set function on the ring $S(V)$ of simple sets generated by V . We shall use the same symbol w to denote that extension. Notice that $w(A) \geq 0$ for all sets $A \in S(V)$ and as a consequence

$$w(A) \leq w(B) \leq w(X) \quad \text{if } A \subset B; A, B \in S(V).$$

Let $c(x)$ and $d(x)$ denote any two Dini derivatives of the function $f(x)$ on the interval (a, b) . We will prove that the functions c and d are equal almost everywhere on X .

The following property follows from the definition of \limsup and \liminf . If $c(x)$ denotes either lower or upper right-sided Dini derivative and $c(x) < u$, then there exists a sequence $h_n > 0$ such that $h_n \rightarrow 0$ and

$$f(x + h_n) - f(x) < u h_n \quad \text{for all } n.$$

Thus denoting by I_n the interval with the end points x and $x + h_n$ we get

$$(25.1) \quad w(I_n) < u v(I_n) \quad \text{for all } n.$$

Similar argument can be applied to the left-sided derivatives yielding the existence of a sequence of intervals I_n with the end points at $x - h_n$ and x such that the estimate (25.1) holds.

Proceeding in a similar manner we can prove that if $d(x)$ denotes any Dini derivative, left or right-sided, upper or lower, and if $s < d(x)$ then there exists a sequence of intervals I_n each of positive lengths and such that $x \in I_n$ and $v(I_n) \rightarrow 0$, when $n \rightarrow \infty$, and

$$s v(I_n) < w(I_n) \quad \text{for all } n.$$

Now consider the set

$$E = \{x \in (a, b) : c(x) < d(x)\}.$$

Since between any two real numbers one can find a pair of rational numbers

$$(25.2) \quad u < s,$$

the set E is equal to the countable union of the sets

$$E_{u,s} = \{x \in (a, b) : c(x) < u < s < d(x)\}.$$

Assuming that the set E is not a null set, will yield a contradiction. Indeed from countable subadditivity of the outer measure v^* we get that at least for one of the sets $E_{u,s}$ we must have

$$0 < t = v^*(E_{u,s}).$$

Take any ε such that $0 < \varepsilon < t/2$ and let G denote an open set containing the set $E_{u,s}$ and such that

$$(25.3) \quad v^*(G) < t + \varepsilon.$$

Since Riemann measure space is regular such an open set exists in accord with Theorem 23.12 on outer regularity of outer measure v^* .

Consider the collection of closed sets

$$(25.4) \quad H = \{I \subset G : I = \emptyset \text{ or } I \cap E_{u,s} \neq \emptyset, 0 < v(I), w(I) < u v(I)\}.$$

It follows from the previous considerations that the collection H forms a Vitali covering of the set $E_{u,s}$.

By Vitali covering theorem there exists a sequence of disjoint intervals I_k forming an interlaced covering of the set $E_{u,s}$ that is

$$E_{u,s} \subset [\bigcup_{k \leq n} I_k] \cup [\bigcup_{k > n} S(I_k)] \quad \text{for all } n.$$

Let J_k denote the interior of the interval I_k . Notice the estimate

$$E_{u,s} \setminus \bigcup_{k \leq n} J_k \subset [\bigcup_{k \leq n} (I_k \setminus J_k)] \cup [\bigcup_{k > n} S(I_k)] \quad \text{for all } n.$$

Since each set $I_k \setminus J_k$ consists of just two points, their union forms a null set. Thus from the preceding set estimate we get

$$v^*(E_{u,s} \setminus \bigcup_{k \leq n} J_k) \leq 5 \sum_{k > n} v(I_k) < \varepsilon$$

for sufficiently large n .

Denote by G_u the open set $\bigcup_{k \leq n} J_k$ and notice the inequality

$$t = v^*(E_{u,s}) \leq v^*(E_{u,s} \cap G_u) + v^*(E_{u,s} \setminus G_u)$$

yielding the estimate

$$(25.5) \quad 0 < t - \varepsilon \leq v^*(E_{u,s} \cap G_u).$$

From the definition of the collection H of sets (25.4) and the fact that G_u forms a simple set from the ring $S(V)$, and the estimate (25.3) on $v^*(G)$, follows that

$$(25.6) \quad \begin{aligned} w(G_u) &= \sum_{k \leq n} w(J_k) = \sum_{k \leq n} w(I_k) \\ &\leq u \sum_{k \leq n} v(I_k) \leq u v(G) \leq u t + u \varepsilon. \end{aligned}$$

Now consider the collection of closed sets

$$(25.7) \quad H_s = \{I \subset G_u : I = \emptyset \text{ or } I \cap E_{u,s} \cap G_u \neq \emptyset, 0 < v(I), s v(I) < w(I)\}.$$

This collection forms a Vitali covering of the set $E_{u,s} \cap G_u$. Let I'_k denote the sequences of disjoint sets forming an interlaced covering of the set. Thus we have

$$E_{u,s} \cap G_u \subset [\bigcup_{k \leq n} I'_k] \cup [\bigcup_{k > n} S(I'_k)] \quad \text{for all } n.$$

Again for sufficiently large index n we get the estimate

$$(25.8) \quad v^*(E_{u,s} \cap G_u) \leq \sum_{k \leq n} v^*(I'_k) + \varepsilon = \sum_{k \leq n} v(I'_k) + \varepsilon.$$

By the lower estimate (25.5) of $v^*(E_{u,s} \cap G_u)$, and the upper estimate (25.8), and the upper estimate (25.6) on $w(G_u)$, we get

$$\begin{aligned} s(t - \varepsilon) &\leq s v^*(E_{u,s} \cap G_u) \leq s \sum_{k \leq n} v(I'_k) + s \varepsilon \\ &\leq \sum_{k \leq n} w(I'_k) + s \varepsilon \leq w(\bigcup_{k \leq n} I'_k) + s \varepsilon \\ &\leq w(G_u) + s \varepsilon \leq u t + u \varepsilon + s \varepsilon \end{aligned}$$

that is

$$s(t - \varepsilon) \leq u t + u \varepsilon + s \varepsilon \quad \text{for all } \varepsilon \in (0, t/2).$$

Thus passing to the limit $\varepsilon \rightarrow 0$ in the above inequality and dividing by t we get $s \leq u$, which contradicts the assumption (25.2) that $u < s$.

Switching the roles of the functions c and d we get that the set where $c(x) > d(x)$ also forms a null set. Thus we have proved that any increasing function f on an interval has a derivative almost everywhere inside of the interval.

Now let us prove that the derivative is finite almost everywhere. Since for an increasing function all the difference quotients are nonnegative, the derivative $f'(x)$ may take on only nonnegative values. So we need to consider only the set

$$E = \{x \in (a, b) : f'(x) = \infty\}.$$

Take any $\varepsilon > 0$ and chose a sufficiently large number $r > 0$ so that

$$w(X)/r \leq \varepsilon/2.$$

Introduce a collection of closed sets

$$H = \{I \in V : I = \emptyset \text{ or } I \cap E \neq \emptyset, 0 < v(I), r v(I) < w(I)\}.$$

The collection H forms a Vitali covering of the set E . Let I_k be a sequence of disjoint intervals forming an interlaced cover of the set E that is

$$E \subset [\bigcup_{k \leq n} I_k] \cup [\bigcup_{k > n} S(I_k)] \quad \text{for all } n.$$

Since

$$\sum_{k > 0} v(I_k) = \sum_{k > 0} v_c(I_k) = v_c(\bigcup_{k > 0} I_k) \leq v_c(X) = f(b) - f(a) < \infty,$$

the above series converges and thus its remainder tends to zero. Choosing sufficiently large index n we can get

$$5 \sum_{k > n} v(I_k) \leq \varepsilon/2.$$

We have the estimate

$$\begin{aligned} v^*(E) &\leq \sum_{k \leq n} v(I_k) + \sum_{k > n} v(S(I_k)) \\ &\leq \frac{1}{r} \sum_{k \leq n} w(I_k) + 5 \sum_{k > n} v(I_k) \\ &\leq \frac{1}{r} w(\bigcup_{k \leq n} I_k) + 5 \sum_{k > n} v(I_k) \\ &\leq \frac{1}{r} w(X) + 5 \sum_{k > n} v(I_k) \leq \varepsilon. \end{aligned}$$

Since ε was fixed but arbitrary this implies $v^*(E) = 0$ that is the set E is a null set.

Hence we have proved that the derivative $f'(x)$ exists and is finite for almost all points x in the interval (a, b) . \square

26. PROPERTIES OF MONOTONE FUNCTIONS

In this section we shall consider further properties of monotone functions. First notice that since a function that is differentiable at a point is continuous at that point, every monotone function on an interval is continuous almost everywhere on that interval.

Theorem 26.1 (Summability of monotone functions). *Let X denote a closed bounded interval $[a, b]$ and (X, V, v) the Riemann measure space on the pre-ring V of all subintervals of X .*

If f is a real-valued monotone function on the interval X then it is Lebesgue summable that is $f \in L(v, R)$.

Proof. We may assume without loss of generality that the function f is increasing. Split the interval $[a, b]$ into two disjoint intervals $I_{1,j}$ of equal length and let $a_{1,j}$ denote the left end of the interval $I_{1,j}$ where $j = 1, 2$. Put

$$s_1(x) = f(a_{1,1})c_{I_{1,1}}(x) + f(a_{1,2})c_{I_{1,2}}(x) \quad \text{for all } x \in X.$$

Notice that

$$s_1(x) \leq f(x) \quad \text{for all } x \in X.$$

Proceeding in similar manner with each of the intervals $I_{1,j}$ by dividing them into two intervals, and proceeding by induction we will obtain an increasing sequence of simple functions

$$s_n(x) = \sum_{j \leq 2^n} f(a_{n,j})c_{I_{n,j}}(x) \leq f(x) \quad \text{for all } x \in X.$$

Notice that the sequence s_n of simple functions is bounded from above by the summable function $f(b)c_X$.

The sequence of values $s_n(x)$ converges at every point of continuity of the function f to the value $f(x)$.

Indeed, take any positive ε and select $\delta > 0$ so that

$$|f(x) - f(y)| < \delta \quad \text{if } y \in (x - \delta, x + \delta) \subset (a, b).$$

Select k so that

$$2^{-n}v(X) < \delta \quad \text{for all } n \geq k.$$

Since each of the intervals $I_{n,j}$ have the same length

$$v(I_{n,j}) = 2^{-n}v(X)$$

and they form a disjoint decomposition of the interval X

$$X = \bigcup_{j \leq 2^n} I_{n,j},$$

we must have that for each $n > k$ there is exactly one interval $I_{n,j}$ containing the point x . Since for each such interval we have

$$I_{n,j} \subset (x - \delta, x + \delta) \quad \text{for all } n > k$$

we see that

$$|s_n(x) - f(x)| = |f(a_{n,j}) - f(x)| < \varepsilon \quad \text{for all } n > k.$$

Thus $s_n(x) \rightarrow f(x)$ at every point of continuity of the function f . It follows from the Monotone Convergence Theorem that the function f belongs to the space $L(v, R)$ of Lebesgue summable functions. \square

26.1. Differentiability of series of monotone functions. For a reference to the following theorem see Fubini [16].

Theorem 26.2 (Fubini). *Let X denote a closed bounded interval $[a, b]$ and (X, V, v) the Riemann measure space on the pre-ring V of all subintervals of X .*

Assume that h_n is a sequence of functions such that all functions of the sequence are either increasing or all are decreasing.

If the series

$$\sum_n h_n(x) = h(x) \quad \text{for all } x \in X$$

converges to a finite value for each x , then there exists a null set D such that the derivative $h'(x)$ exists and we have the equality

$$\sum_n h'_n(x) = h'(x) \quad \text{for all } x \in X \setminus D.$$

Proof. Without loss of generality we may assume that all functions of the sequence are increasing and $h_n(a) = 0$ for all n . Assume that for each n the derivative $h'_n(x)$ exists for $x \in X \setminus D_n$ where D_n is a null set. Since the function h as a sum of a series of increasing functions is itself increasing the derivative $h'(x)$ exists for all $x \notin D_0$, where D_0 is a null set. Let D be the null set representing the union

$$D = \bigcup_{n \geq 0} D_n.$$

Take any point $x \in X \setminus D$ and consider any point $y \in X$ such that $x < y$. Since the corresponding difference quotients of functions h_n are all nonnegative, we must have the estimate

$$\sum_{n \leq m} \frac{h_n(y) - h_n(x)}{y - x} \leq \frac{h(y) - h(x)}{y - x} \quad \text{for all } m.$$

Thus for a fixed index m passing to the limit $y \rightarrow x$ we get

$$\sum_{n \leq m} h'_n(x) \leq h'(x) \quad \text{for all } m \quad \text{and } x \notin D.$$

Since all terms $h'_n(x)$ are nonnegative and the above partial sums are bounded we must have

$$\sum_{n=1}^{\infty} h'_n(x) \leq h'(x) \quad \text{for all } x \notin D.$$

The above implies

$$h'_n(x) \rightarrow 0 \quad \text{for all } x \notin D.$$

Introduce notation for the partial sums

$$s_n = h_1 + h_2 + \cdots + h_n \quad \text{for all } n$$

and select a subsequence s_{k_n} so that

$$|s_{k_n}(b) - h(b)| \leq 2^{-n} \quad \text{for all } n.$$

Notice that the series with terms $f_n = (h - s_{k_n})$ consists of increasing functions and it converges for all $x \notin D$ to a finite function. Thus from the previous considerations we can conclude that

$$f'_n(x) = h'(x) - s'_{k_n}(x) \rightarrow 0 \quad \text{for all } x \notin D.$$

Since the sequence of partial sums s_n is increasing the above relation implies the convergence

$$s'_n(x) \rightarrow h'(x) \quad \text{for all } x \notin D,$$

which is equivalent to

$$\sum_{n=1}^{\infty} h'_n(x) = h'(x) \quad \text{for all } x \notin D.$$

□

Corollary 26.3 (Fubini). *Let $I \subset \mathbb{R}$ be any interval of reals. Assume that f_n is a sequence of increasing functions on the interval I .*

If the sequence f_n increasingly converges to a finite-valued function f at every point of the interval I , then the function f is differentiable almost everywhere on the interval I and moreover the derivatives $f'_n(x)$ increasingly converge to the derivative $f'(x)$ almost everywhere on I .

27. INVARIANT MEASURES INDUCE INVARIANT INTEGRALS

Definition 27.1 (Invariant measure). *Let X be an abstract space and V a prering of subsets of X . Assume that the triple (X, V, v) forms a positive measure space and T represents a map from the space X into X .*

*We shall say that the measure space (X, V, v) is **invariant** under the map T if*

$$T^{-1}(A) \in V \quad \text{and} \quad v(T^{-1}(A)) = v(A) \quad \text{for all } A \in V.$$

The Riemann measure space (\mathbb{R}, V, v) over the reals \mathbb{R} is invariant under any translation map

$$T(x) = x + h \quad \text{for all } x \in \mathbb{R},$$

since a translation operation maps intervals onto intervals and preserves the length of the interval. The Riemann measure space is also invariant under the reflection $x \mapsto -x$. The same is true for Riemann measure space (\mathbb{R}^n, V, v) .

The counting measure space (X, V, v) is invariant under any map T of X onto X which is one-to-one.

Theorem 27.2 (Invariant integral). *Let X be an abstract space and V a prering of subsets of X . Assume that the triple (X, V, v) forms a positive measure space and Y a Banach space. Assume that T is a map from the space X into X .*

If the measure space (X, V, v) is invariant under the map T , then the operator

$$f \mapsto f \circ T \quad \text{for all } f \in L(v, Y)$$

maps the space $L(v, Y)$ of Bochner summable functions into $L(v, Y)$ and preserves the integral, that is,

$$\int f \, dv = \int f \circ T \, dv \quad \text{for all } f \in L(v, Y).$$

Proof. Notice that the theorem is valid for simple functions. Indeed if $A \in V$ then for the characteristic function of A we have

$$c_A \circ T(x) = 1 \iff T(x) \in A \iff x \in T^{-1}(A) \iff c_{T^{-1}(A)}(x) = 1.$$

Thus for a simple function

$$s = y_1 c_{A_1} + \cdots y_k c_{A_k}$$

we get

$$s \circ T = y_1 c_{T^{-1}(A_1)} + \cdots y_k c_{T^{-1}(A_k)}$$

and therefore

$$\begin{aligned} \int s \circ T \, dv &= y_1 \int c_{T^{-1}(A_1)} \, dv + \cdots + y_k \int c_{T^{-1}(A_k)} \, dv \\ &= y_1 v(T^{-1}(A_1)) + \cdots + y_k v(T^{-1}(A_k)) \\ &= y_1 v(A_1) + \cdots + y_k v(A_k) = \int s \, dv. \end{aligned}$$

As a consequence the transformation $f \mapsto f \circ T$ maps a basic sequence into a basic sequence. So take any Bochner summable function $f \in L(v, Y)$. By definition of the space $L(v, Y)$ there exists a basic sequence $s_n \in S(V, Y)$ and a null set A such that

$$s_n(x) \rightarrow f(x) \quad \text{when } x \notin A.$$

Thus

$$s_n \circ T(x) \rightarrow f \circ T(x) \quad \text{when } T(x) \notin A$$

or equivalently

$$s_n \circ T(x) \rightarrow f \circ T(x) \quad \text{when } x \notin T^{-1}(A).$$

Clearly the sequence $s_n \circ T$ is also basic. Since the map $A \mapsto T^{-1}(A)$ considered as a map from the power set $\mathcal{P}(X)$ into itself, is monotone and maps unions of sets into unions of their images, the set $T^{-1}(A)$ is a null set. Thus we must have that $f \circ T \in L(v, Y)$ and

$$\int f \circ T \, dv = \lim_n \int s_n \circ T \, dv = \lim_n \int s_n \, dv = \int f \, dv.$$

□

28. INTEGRATION OVER THE SPACE R OF REALS

If (R, V, v) is the Riemann measure space and X any subinterval of R then let V denote the prering of all bounded subintervals of X , and $v(A)$ the length of the interval $A \subset X$. Clearly the space (X, V, v) is a measure space as a subspace of the Riemann measure space. We shall call this measure space (X, V, v) the *Riemann measure space over the interval X* .

For the case of a Riemann measure space over an interval X we shall use the customary notation for the integral of a Bochner summable function $f \in L(v, Y)$. We shall write

$$\begin{aligned} \int_{t_1}^{t_2} f(t) dt &= \int c_{[t_1, t_2]} f dv \quad \text{if } t_1 \leq t_2, \ t_1, t_2 \in X, \\ \int_{t_1}^{t_2} f(t) dt &= - \int c_{[t_2, t_1]} f dv \quad \text{if } t_1 > t_2, \ t_1, t_2 \in X. \end{aligned}$$

Adopting the above notation yields a convenient formula for any $f \in L(v, Y)$

$$\int_{t_1}^{t_2} f(t) dt + \int_{t_2}^{t_3} f(t) dt + \int_{t_3}^{t_1} f(t) dt = 0 \quad \text{for all } t_1, t_2, t_3 \in X.$$

Definition 28.1 (Indefinite integral). *Let (X, V, v) denote a Riemann measure space over an interval X and Y a Banach space. Let $f \in L(v, Y)$ be a Bochner summable function.*

*By an **indefinite integral** of the function f we shall understand any function of the form*

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X \text{ and some } a \in X.$$

We shall prove later a theorem due to Lebesgue that indefinite integral F of a function f , summable with respect to Riemann measure, has a derivative almost everywhere and its derivative F' coincides with the function f almost everywhere, but first notice the following simple consequence of continuity of the integrand f .

Proposition 28.2 (Differentiability at continuity points). *Let X be an interval and let (X, V, v) denote the Riemann measure space over X and Y any Banach space.*

Let $f \in L(v, Y)$ be a Bochner summable function and let F denote its indefinite integral

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in I.$$

If the function f is continuous at a point $p \in I$, then the function F is differentiable at p and $F'(p) = f(p)$.

Proof. Take any $\varepsilon > 0$ and select $\delta > 0$ so that

$$|f(p+h) - f(p)| \leq \varepsilon \quad \text{if } |h| \leq \delta \quad \text{and } x, x+h \in X.$$

Then we have

$$\left| \frac{F(p+h) - F(p)}{h} - f(p) \right| = \left| \frac{1}{h} \int_p^{p+h} (f(t) - f(p)) dt \right| \leq \varepsilon$$

if $0 < |h| \leq \delta$. Thus $F'(p) = f(p)$. \square

Theorem 28.3 (Lipschitzian property of indefinite integral). *Let (I, V, v) denote the Riemann measure space and Y a Banach space. Assume that f is a Bochner summable function on the interval I and F its indefinite integral.*

If for some constant M we have $|f(x)| \leq M$ almost everywhere on I then

$$|F(x) - F(y)| \leq M|x - y| \quad \text{for all } x, y \in I,$$

that is the function F is Lipschitzian on the interval I .

Proof. The proof is straightforward and we leave it to the reader. \square

Definition 28.4 (Local summability). *Assume that (X, V, v) is a positive measure space over a prering V of an abstract space X .*

*A function f from X into a Banach space Y is said to be **locally summable** if for every set $A \in V$ the function $c_A f$ belongs to the space $L(v, Y)$ of Bochner summable functions.*

Clearly every summable function $f \in L(v, Y)$ is locally summable on X but if the set X is not in the prering V then function c_X is locally summable on X but it does not have to be summable on X .

Consider for example an infinite set X and the counting measure, or the space R of reals and the Riemann measure on the prering of all bounded intervals.

Definition 28.5 (Newton's formula). *Assume that F is a function from an interval $I \subset R$ into a Banach space Y . We shall say that the function F satisfies **Newton's formula** on the interval I if the derivative $F'(x)$ exists for almost all $x \in I$, and it is locally Bochner summable on I , and*

$$F(x_1) - F(x_2) = \int_{x_2}^{x_1} F'(t) dt \quad \text{for all } x_1, x_2 \in I.$$

Theorem 28.6 (Newton's formula and Lipschitzian functions). *Let (I, V, v) be the Riemann measure space over an interval I . Assume that F is a Lipschitzian function from the interval I into a Banach space Y .*

If the derivative $F'(x)$ exists for almost all $x \in I$ then the derivative is locally summable on I and Newton's formula holds

$$F(x_1) - F(x_2) = \int_{x_2}^{x_1} F'(t) dt \quad \text{for all } x_1, x_2 \in I.$$

Proof. In the case when the interval I is bounded on the right, notice that the Lipschitz condition implies the existence of the limit

$$\lim_{x \rightarrow b} F(x) = y_0$$

where b is the right end of the interval I . In such a case extend the function F to the right by the formula

$$F(x) = y_0 \quad \text{for all } x \geq b.$$

This operation will not change the Lipschitzian property nor the differentiability almost everywhere of the function F .

Take any sequence h_n of positive numbers converging to zero and consider the sequence of functions

$$f_n(x) = \frac{1}{h_n}(F(x + h_n) - F(x)) \quad \text{for all } x \in I.$$

Notice that the functions f_n are well defined and are continuous on the interval I and as such they are locally Bochner summable on I . If M denotes the Lipschitz constant of F then

$$|f_n(x)| = \frac{1}{h_n}|F(x + h_n) - F(x)| \leq M \quad \text{for all } x \in I.$$

Moreover by assumption we have $\lim_n f_n(x) = F'(x)$ for almost all $x \in I$. Thus by Lebesgue's Dominated Convergence Theorem the function F' is locally summable on the interval I and

$$\int_{x_1}^{x_2} F'(t) dt = \lim_n \int_{x_1}^{x_2} f_n(t) dt \quad \text{for all } x_1 \leq x_2, x_1, x_2 \in I.$$

From linearity of the integral and the fact that with respect to Riemann measure the integral is invariant under translations, we have

$$\begin{aligned} \int_{x_1}^{x_2} f_n(t) dt &= \frac{1}{h_n} \left[\int_{x_1}^{x_2} F(t + h_n) dt - \int_{x_1}^{x_2} F(t) dt \right] \\ &= \frac{1}{h_n} \left[\int_{x_1+h_n}^{x_2+h_n} F(t) dt - \int_{x_1}^{x_2} F(t) dt \right] \\ &= \frac{1}{h_n} \left[\left(\int_{x_1+h_n}^{x_2} F(t) dt + \int_{x_2}^{x_2+h_n} F(t) dt \right) \right. \\ &\quad \left. - \left(\int_{x_1}^{x_1+h_n} F(t) dt + \int_{x_1+h_n}^{x_2} F(t) dt \right) \right] \\ &= \frac{1}{h_n} \left[\int_{x_2}^{x_2+h_n} F(t) dt - \int_{x_1}^{x_1+h_n} F(t) dt \right] \end{aligned}$$

By assumption the function F is Lipschitzian, so it is continuous. Passing to the limit $n \rightarrow \infty$ in the above equality we get

$$\int_{x_1}^{x_2} F'(t) dt = F(x_2) - F(x_1) \quad \text{for all } x_1 \leq x_2, x_1, x_2 \in I.$$

For the case when the integration limits are in reversed order $x_2 < x_1$, from the above formula and the definition of the integral with respect to Riemann measure follows that Newton's formula holds for all $x_1, x_2 \in I$. \square

Newton's formula in the case when the derivative F' is continuous is known as the **Fundamental Theorem of Calculus**. The existence of the derivative F' even at every point of the interval does not guarantee the validity of Newton's formula.

Consider for instance the real-valued function defined by the formulas

$$(28.1) \quad F(x) = x^2 \sin(1/x^2) \quad \text{for all } x \neq 0 \quad \text{and} \quad F(0) = 0.$$

The derivative F' exists at every point $x \in R$ but it is not summable on any interval containing in its interior the point $x = 0$. One can prove this fact by showing that the function F is not *absolutely continuous*, which is a necessary condition for a function to be an indefinite integral of a locally summable function. So let us consider this notion.

Definition 28.7 (Absolute continuity). *Assume that (X, V, v) is a Riemann measure space over an interval X and Y is a Banach space.*

*A function $F : X \rightarrow Y$ is said to be **absolutely continuous** on X if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite system of disjoint intervals*

$$I_k = (a_k, b_k) \subset X \quad \text{where} \quad k = 1, 2, \dots, n$$

we have

$$\sum_k |F(b_k) - F(a_k)| < \varepsilon \quad \text{if} \quad \sum_k v(I_k) < \delta.$$

Notice that if F is a Lipschitzian function with constant M then the formula for the number δ corresponding to $\varepsilon > 0$ in the above definition is

$$\delta = \varepsilon/M.$$

Thus every Lipschitzian function is absolutely continuous. We have the following general theorem.

Theorem 28.8 (Absolute continuity of indefinite integral). *Assume that (X, V, v) is a Riemann measure space over an interval X and Y is a Banach space.*

If $f \in L(v, Y)$ then its indefinite integral given by

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X \quad \text{and some } a \in X$$

is absolutely continuous.

Proof. Take any $\varepsilon > 0$. By Lemma 16.3 on density of simple functions in the space $L(v, Y)$ there exists a function $s \in S(V, Y)$ such that

$$\|f - s\| < \varepsilon/2.$$

Let S denote the indefinite integral of the function s

$$S(x) = \int_a^x s(t) dt \quad \text{for all } x \in X$$

and G the indefinite integral of the function $g = f - s$.

Let $M = \sup \{|s(x)| : x \in X\}$. Since M represents a Lipschitz constant of S we have for any system of disjoint intervals

$$I_k = (a_k, b_k) \quad \text{where} \quad k = 1, 2, \dots, n$$

and $\delta = \varepsilon/(2M)$ that

$$\sum_k |S(b_k) - S(a_k)| < \varepsilon/2 \quad \text{if} \quad \sum_k v(I_k) < \delta.$$

Put $A = \bigcup_k I_k$ and notice the estimate

$$\sum_k |G(b_k) - G(a_k)| \leq \sum_k \int_{a_k}^{b_k} |g(t)| dt \leq \int_A |g(t)| dt \leq \|g\| < \varepsilon/2.$$

Thus we have

$$\sum_k |F(b_k) - F(a_k)| \leq \sum_k |G(b_k) - G(a_k)| + \sum_k |S(b_k) - S(a_k)| < \varepsilon.$$

□

We leave it to the reader to show that the function F given by the formula (28.1) is not absolutely continuous in any interval containing the point $x = 0$ and thus it is not an indefinite integral of a Bochner summable function in any such interval.

Our goal is to prove that any function F from an interval X into a Banach space Y possessing almost everywhere on X a locally summable derivative F' satisfies Newton's formula

$$F(y) - F(x) = \int_x^y F'(t) dt \quad \text{for all } x, y \in X$$

if and only if the function F is absolutely continuous. To accomplish this we will need several theorems of Lebesgue concerning this topic for real-valued functions.

29. LEBESGUE THEORY OF ABSOLUTELY CONTINUOUS FUNCTIONS

We start this section by proving several important lemmas and propositions playing a key role in the development of the Lebesgue theory of absolutely continuous functions.

Proposition 29.1. *Let (X, V, v) be any measure space over a pre-ring V of an abstract space X .*

If for some summable function $f \in L(v, R)$ we have

$$\int_A f(t) dt = 0 \quad \text{for all } A \in V$$

then $f(x) = 0$ for almost all $x \in X$.

Proof. By assumption of the theorem for any set $A \in V$ we have

$$\int c_A f dv = 0.$$

As a consequence for any simple function s we must have

$$\int s f dv = 0 \quad \text{for all } s \in S(V, R).$$

Since simple functions are dense in the Lebesgue space $L(v, R)$, for any $\varepsilon > 0$ there is a simple function $s_0 \in S(V, R)$ such that

$$\|s_0 - f\| < \varepsilon.$$

If the simple function s_0 has representation

$$s_0 = \sum_k r_k c_{A_k},$$

where $\{A_k\}$ is a finite family of disjoint sets from the prering V , define function s by the formula

$$s = \sum_k \text{sign}(r_k) c_{A_k},$$

where

$$\text{sign}(r) = r/|r| \quad \text{if} \quad r \neq 0 \quad \text{and} \quad \text{sign}(r) = 0 \quad \text{if} \quad r = 0.$$

By linearity and monotonicity of the integral we have

$$\begin{aligned} \|s_0\| &= \int |s_0| dv = \int s s_0 dv = \int s(s_0 - f) dv + \int s f dv \\ &\leq \int |s_0 - f| dv + 0 = \|s_0 - f\| < \varepsilon. \end{aligned}$$

Thus from the triangle inequality for a norm we get

$$\|f\| \leq \|s_0\| + \|s_0 - f\| < 2\varepsilon.$$

Since ε was fixed but arbitrary we must have $f(x) = 0$ for almost all $x \in X$. \square

The above proposition can be immediately generalized to the case of Bochner summable functions but first let us prove a lemma that is of interest by itself.

29.1. Commutativity of a linear bounded operator with integral.

Lemma 29.2. *Let (X, V, v) be any measure space over a prering V of an abstract space X . Assume that Y, Z are Banach spaces and u a linear bounded operator from Y to Z .*

If $f \in L(v, Y)$ then $u \circ f \in L(v, Z)$ and

$$u\left(\int f dv\right) = \int u \circ f dv$$

Proof. By definition of a Bochner summable function $f \in L(v, Y)$ there exists a basic sequence s_n and a null set D such that

$$s_n(x) \rightarrow f(x) \quad \text{for all} \quad x \in X \setminus D.$$

Since the operator u is linear and bounded, the composition $S_n(x) = u(s_n(x))$ yields a basic sequence converging to the function $u(f(x))$ for every $x \in X \setminus D$. Thus we have $u \circ f \in L(v, Z)$ and by linearity and continuity of the operator u we get

$$\int u \circ f dv = \lim_n \int u \circ s_n dv = u\left(\lim_n \int s_n dv\right) = u\left(\int f dv\right).$$

\square

Proposition 29.3. *Let (X, V, v) be any measure space over a prepring V of an abstract space X and let Y be a Banach space.*

If for some Bochner summable function $f \in L(v, Y)$ we have

$$\int_A f(t) dt = 0 \quad \text{for all } A \in V$$

then $f(x) = 0$ for almost all $x \in X$.

Proof. Let Y' denote the dual Banach space of the space Y . It follows from the assumption of the theorem that for any set $A \in V$ and any functional $y' \in Y'$ we have

$$0 = \int y' \circ (c_A f) dv = \int c_A y' \circ (f) dv.$$

As a consequence for any simple function $s \in S(V, Y')$ we must have

$$\int u(s, f) dv = 0 \quad \text{for all } s \in S(V, Y'),$$

where u denotes the bilinear bounded operator

$$u(y', y) = y'(y) \quad \text{for all } y' \in Y' \quad \text{and } y \in Y.$$

Since simple functions are dense in the Lebesgue space $L(v, Y)$, for any $\varepsilon > 0$ there is a simple function $s_0 \in S(V, Y)$ such that

$$\|s_0 - f\| < \varepsilon.$$

If the simple function s_0 has representation

$$s_0 = \sum_k y_k c_{A_k},$$

where $\{A_k\}$ is a finite family of disjoint sets from the prepring V , define function s by the formula

$$s = \sum_k y'_k c_{A_k},$$

where $y'_k \in Y'$ is linear functional such that

$$y'_k(y_k) = |y_k| \quad \text{and} \quad |y'_k| = 1..$$

By linearity and monotonicity of the integral we have

$$\begin{aligned} \|s_0\| &= \int |s_0| dv = \int u(s, s_0) dv = \int u(s, (s_0 - f)) dv + \int u(s, f) dv \\ &\leq \int |s_0 - f| dv + 0 = \|s_0 - f\| < \varepsilon. \end{aligned}$$

Thus from the triangle inequality for a norm we get

$$\|f\| \leq \|s_0\| + \|s_0 - f\| < 2\varepsilon.$$

Since ε was fixed but arbitrary we must have $f(x) = 0$ for almost all $x \in X$. \square

Lemma 29.4. *Let (X, V, v) be the Riemann measure space over a closed bounded interval $X = [a, b]$.*

If for some Lebesgue summable function $f \in L(v, R)$ we have

$$\int_a^x f(t) dt = 0 \quad \text{for all } x \in X$$

then $f(x) = 0$ for almost all $x \in X$.

Proof. It follows from the assumption of the theorem and linearity of the integral, that for any interval $A \in V$ we must have

$$\int c_A(t) f(t) dt = 0 \quad \text{for all } A \in V.$$

Thus from the preceding proposition we get $f(x) = 0$ for almost all $x \in X$. □

Lemma 29.5. *Let (X, V, v) be the Riemann measure space over a closed bounded interval $X = [a, b]$.*

If f is a nonnegative bounded function such that $f \in L(v, R)$ then its indefinite integral F given by

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X$$

is differentiable almost everywhere on X and

$$F'(x) = f(x) \quad \text{for almost all } x \in X.$$

Proof. Since the function F is increasing on the interval X it is differentiable almost everywhere. Denote its derivative by g . Thus $F'(x) = g(x)$ for almost all $x \in X$.

Since the function f by assumption is bounded its indefinite integral F is Lipschitzian. Since we already established that for such a function F the Newton formula is valid, we have

$$F(x) - F(a) = \int_a^x F'(t) dt = \int_a^x g(t) dt \quad \text{for all } x \in X.$$

Since $F(a) = 0$ we must have

$$\int_a^x g(t) dt = F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X.$$

That is

$$\int_a^x (g(t) - f(t)) dt = 0 \quad \text{for all } x \in X.$$

By preceding Lemma and the above we must have $(g(x) - f(x)) = 0$ almost everywhere and therefore

$$F'(x) = g(x) = f(x) \quad \text{for almost all } x \in X.$$

□

The following proposition is just a stepping stone to a more general result that will follow.

Proposition 29.6 (Lebesgue). *Let (X, V, v) be the Riemann measure space over a closed bounded interval $X = [a, b]$.*

If f is a nonnegative Lebesgue summable function $f \in L(v, R)$ then its indefinite integral F given by

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X$$

is differentiable almost everywhere on X and

$$F'(x) = f(x) \quad \text{for almost all } x \in X.$$

Proof. Since $X \in V$ the functions of the form nc_X are simple and as such they are summable. Since the space $L(v, R)$ forms a linear lattice we must have

$$f_n = f \wedge nc_X \in L(v, R) \quad \text{for all } n = 1, 2, \dots$$

Notice that the sequence f_n increasingly converges everywhere on the set X to the function f .

Introduce the indefinite integrals

$$F_n(x) = \int_a^x f_n(t) dt \quad \text{and} \quad F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X.$$

Since the above indefinite integrals form increasing functions, their derivatives $F'_n(x)$ and $F'(x)$ exist for almost all $x \in X$.

Since each function f_n is summable and bounded, from the preceding lemma we have $F'_n(x) = f_n(x)$ for almost all $x \in X$.

It follows from the Monotone Convergence Theorem, that the sequence F_n increasingly converges everywhere on X to the function F . Thus by Corollary to Fubini theorem 26.3 we have that the derivative F' exists almost everywhere on X and

$$f_n(x) = F'_n(x) \rightarrow F'(x) \quad \text{for almost all } x \in X.$$

Thus by Lebesgue's Dominated Convergence Theorem we have

$$\int_a^x f_n(t) dt \rightarrow \int_a^x F'(t) dt \quad \text{for all } x \in X$$

and also

$$\int_a^x f_n(t) dt \rightarrow \int_a^x f(t) dt \quad \text{for all } x \in X.$$

From the uniqueness of the limit we must have

$$\int_a^x F'(t) dt = \int_a^x f(t) dt \quad \text{for all } x \in X,$$

or equivalently

$$\int_a^x (F'(t) - f(t)) dt = 0 \quad \text{for all } x \in X,$$

which implies $F'(x) = f(x)$ for almost all $x \in X$. □

Definition 29.7 (Lebesgue points of a summable function). *Let (X, V, v) be any Riemann measure space over an interval X and Y a Banach space. Assume that f is a locally Bochner summable function from X into Y .*

*A point $p \in X$ is called a **Lebesgue point** of the function f if*

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - f(p)| dt = 0.$$

The proof of the following theorem is due to Lebesgue who proved it for the case of real-valued functions. It carries over to Bochner summable functions with only minor modifications.

Theorem 29.8. *Let (X, V, v) be any Riemann measure space over an interval X and Y a Banach space.*

If f is a locally Bochner summable function from X into Y , then almost every point of X is a Lebesgue point of the function f .

Proof. Without loss of generality we may assume that the set X forms a closed bounded interval. Thus the function f is summable on X . Therefore there exists a basic sequence $s_n \in S(V, Y)$ of simple functions and a null set D_0 such that

$$s_n(x) \rightarrow f(x) \quad \text{for all } x \in X \setminus D_0.$$

Notice that the image set $s_n(X)$ is finite for each n and thus the set $B = \bigcup_n s_n(X)$ is at most countable.

Represent the set B as a sequence

$$B = \{y_1, y_2, \dots\}.$$

It follows from the triangle inequality for the norm that for any index m the map $y \mapsto |y - y_m|$ is Lipschitzian and as such it maps a basic sequence into a basic sequence. Thus the sequence $|s_n(x) - y_m|$ for fixed m is basic and it converges almost everywhere on X to the function $|f(x) - y_m|$. Therefore each such function belongs to the space $L(v, R)$.

By Lebesgue's Proposition 29.6 there exist null sets D_m such that

$$|f(p) - y_m| = \lim_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - y_m| dt \quad \text{for all } p \in X \setminus D_m.$$

Put $D = \bigcup_{m \geq 0} D_m$. For any $p \in X \setminus D$ and any index m we have

$$\begin{aligned} & \limsup_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - f(p)| dt \\ & \leq \limsup_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - y_m| dt + \limsup_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |y_m - f(p)| dt \\ & = |f(p) - y_m| + |y_m - f(p)| = 2|f(p) - y_m|. \end{aligned}$$

Taking any $\varepsilon > 0$ and choosing m so that $|f(p) - y_m| < \varepsilon/2$ we get

$$\limsup_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - f(p)| dt < \varepsilon$$

that is

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_p^{p+h} |f(t) - f(p)| dt = 0.$$

Hence the proof is complete. \square

The following theorem is a simple consequence of the preceding one.

Theorem 29.9 (Differentiability a.e. of indefinite integral). *Let (X, V, v) be any Riemann measure space over an interval X and Y a Banach space.*

If f is a locally Bochner summable function from X into Y , then its indefinite integral F defined by the formula

$$F(x) = \int_a^x f(t) dt \quad \text{for all } x \in X \quad \text{and some } a \in X$$

is differentiable almost everywhere on X and we have the equality

$$F'(x) = f(x) \quad \text{for almost all } x \in X.$$

Proof. Let $p \in X$ be a Lebesgue point of the function f . Take any number $h \neq 0$ such that $p + h \in X$ and form the difference quotient of the function F . We have the following estimate

$$\begin{aligned} \left| \frac{F(p+h) - F(p)}{h} - f(p) \right| &= \left| \frac{1}{h} \int_p^{p+h} f(t) dt - f(p) \right| \\ &= \left| \frac{1}{h} \int_p^{p+h} [f(t) - f(p)] dt \right| \\ &\leq \frac{1}{h} \int_p^{p+h} |f(t) - f(p)| dt. \end{aligned}$$

The last expression in the above estimates by definition of a Lebesgue point converges to zero when h tends to zero. Thus $F'(p) = f(p)$ for almost all $p \in X$. \square

29.2. Absolutely continuous function with zero derivative.

Theorem 29.10. *Let (X, V, v) be a Riemann measure space over a closed bounded interval $X = [a, b]$ on the pre-ring V of all subintervals of X and Y a Banach space.*

If F is an absolutely continuous function from X into Y and its derivative $F'(x)$ exist for almost all $x \in X$ and is equal to zero almost everywhere on X then the function F is constant on the interval X .

Proof. Define a set function η on the prering V by the formula

$$\eta(A) = F(\beta) - F(\alpha) \quad \text{for all } A \in V,$$

where $\alpha \leq \beta$ are the end points of the interval A . Notice that the set function η is finitely additive on the prering V and as such it can be uniquely extended onto the ring $S(V)$ of simple sets generated by V .

We shall prove that $F(a) = F(c)$ for any point $c \in [a, b]$. To this end consider the set

$$E = \{x \in [a, c] : F'(x) = 0\}.$$

Take any $\varepsilon > 0$ and introduce the family of closed intervals

$$H = \{I \subset [a, c] : I = \emptyset \text{ or } I \cap E \neq \emptyset, 0 < v(I), |\eta(I)| < \varepsilon v(I)\}$$

The family H forms a Vitali covering of the set E . Thus there exists a sequence of disjoint intervals $I_j \in H$ forming an interlaced cover of the set E that is

$$E \subset [\bigcup_{j \leq n} I_j] \cup [\bigcup_{j > n} S(I_j)] \quad \text{for all } n.$$

Since the set $[a, c] \setminus E$ is a null set, there exists a sequence of intervals J_j such that $\sum_j v(J_j) < 1$ and

$$[a, c] \setminus E \subset \bigcup_{j > n} J_j \quad \text{for all } n.$$

Thus we have the relation

$$[a, c] \subset [\bigcup_{j \leq n} I_j] \cup [\bigcup_{j > n} S(I_j)] \cup [\bigcup_{j > n} J_j] \quad \text{for all } n.$$

The above yields

$$A_n = [a, c] \setminus [\bigcup_{j \leq n} I_j] \subset [\bigcup_{j > n} S(I_j)] \cup [\bigcup_{j > n} J_j] \quad \text{for all } n.$$

Notice that each set A_n belongs to the ring $S(V)$ of simple sets and from countable subadditivity of the measure v_c we have

$$\begin{aligned} v_c(A_n) &\leq \sum_{j > n} v_c(S(I_j)) + \sum_{j > n} v_c(J_j) \\ &\leq 5 \sum_{j > n} v(I_j) + \sum_{j > n} v(J_j) \quad \text{for all } n \end{aligned}$$

implying $v_c(A_n) \rightarrow 0$. So for sufficiently large n we must have $v_c(A_n) < \delta$, where δ is selected by absolute continuity of the function F so that for every disjoint finite collection of intervals $B_k \subset [a, c]$ we have

$$\sum_k |\eta(B_k)| < \varepsilon \quad \text{if } \sum_k v(B_k) < \delta.$$

Let $B_k \in V$ ($k = 1, 2, \dots, m$) be a finite decomposition of the set A_n . Then the family of intervals

$$\{B_k : k \leq m\} \cup \{I_j : j \leq n\}$$

represents a finite disjoint decomposition of the interval $[a, c]$. Thus by finite additivity of the set function η we have

$$\eta([a, c]) = \sum_{k \leq m} \eta(B_k) + \sum_{j \leq n} \eta(I_j)$$

yielding

$$\begin{aligned} |F(c) - F(a)| &= |\eta([a, c])| \\ &\leq \sum_{k \leq m} |\eta(B_k)| + \varepsilon \sum_{j \leq n} v(I_j) \\ &\leq \varepsilon + \varepsilon v([a, c]) = \varepsilon(1 + c - a). \end{aligned}$$

Since ε was fixed but arbitrary we must have

$$F(c) = F(a) \quad \text{for all } c \in [a, b].$$

□

29.3. Newton's formula and absolute continuity.

Theorem 29.11. *Let (X, V, v) be a Riemann measure space over a closed bounded interval $X = [a, b]$ on the pre-ring V of all subintervals of X and Y a Banach space.*

Assume that a function F from X into Y has a derivative $F'(x)$ for almost all $x \in X$ and the derivative is Bochner summable that is $F' \in L(v, Y)$. Then the function F satisfies the Newton formula

$$\int_x^y F'(t) dt = F(y) - F(x) \quad \text{for all } x, y \in X$$

if and only if the function F is absolutely continuous on X .

Proof. To prove the necessity of the condition take any function F and assume that it satisfies the Newton formula. Thus we have

$$F(x) = F(a) + \int_a^x F'(t) dt \quad \text{for all } x \in X.$$

Since we proved before that the indefinite integral of a Bochner summable function is absolutely continuous, from the above formula we can deduce function F must be absolutely continuous.

Conversely, assuming that the function F is absolutely continuous on X introduce function

$$H(x) = F(x) - \int_a^x F'(t) dt \quad \text{for all } x \in X.$$

The function H as a difference of two absolutely continuous functions is absolutely continuous on X . Notice that $H'(x) = F'(x) - F'(x) = 0$ for almost all $x \in X$. Thus by Theorem 29.10 on absolutely continuous function whose derivative is equal to zero almost everywhere, we get that H is a constant function on the interval X . Therefore

$$H(x) = H(a) = F(a) \quad \text{for all } x \in X$$

which implies

$$F(x) - F(a) = \int_a^x F'(t) dt \quad \text{for all } x \in X$$

and the above implies the Newton formula

$$F(y) - F(x) = \int_x^y F'(t) dt \quad \text{for all } x, y \in X.$$

□

29.4. A consequences of Lebesgue-Bochner theory. In Calculus courses we define an **antiderivative** F of a continuous function f as any function such that $F' = f$ and we prove the Newton formula which is also called the *Fundamental Theorem of Calculus*.

Let I be any interval and Y a Banach space. Denote by L the family of all functions

$$f : I \mapsto Y$$

that are locally Bochner summable on I with respect to Riemann measure space (I, V, v) . This family contains all continuous functions from I into the space Y .

The above theorem shows that we can extend the notion of an antiderivative of a function $f \in L$ by considering functions F that are absolutely continuous on every bounded subinterval of the interval I and such that $F' = f$ almost everywhere on the interval I .

Such antiderivatives satisfy Newton's formula. Moreover every function $f \in L$ possesses such an antiderivative and any two antiderivatives of the same function differ by a constant. This is one of the important consequences of Lebesgue-Bochner theory of summable functions.

30. INTEGRATION BY PARTS FOR LEBESGUE-BOCHNER SUMMABLE FUNCTIONS

Theorem 30.1 (Integration by parts). *Let X denote a closed bounded interval $[a, b]$ and assume that (X, V, v) represents the Riemann measure space.*

Let Y be a Banach space and Z the field of reals R or the field C of complex numbers considered as a Banach space depending on whether the Banach space Y is over the field R or C .

*If $f \in L(v, Z)$ and $g \in L(v, Y)$ and F and G are absolutely continuous functions on X such that $F' = f$ and $G' = g$ almost everywhere on X then the following formula known as **integration by parts** is true*

$$\int_a^b f(t)G(t) dt + \int_a^b F(t)g(t) dt = F(b)G(b) - F(a)G(a).$$

Proof. Since the functions F and G are absolutely continuous on the closed bounded interval, they are bounded. Let $\|F\|$ and $\|G\|$ denote the supremum norm on X .

Put $H = FG$. Take any finite family of disjoint intervals $I_j = [a_j, b_j]$. We have the estimate

$$\begin{aligned} |H(b_j) - H(a_j)| &= |F(b_j)G(b_j) - F(b_j)G(a_j) + F(b_j)G(a_j) - F(a_j)G(a_j)| \\ &\leq \|F\| |G(b_j) - G(a_j)| + \|G\| |F(b_j) - F(a_j)|. \end{aligned}$$

Thus from the above estimate and from absolute continuity of the functions F and G follows the absolute continuity of their product H .

Notice that the function H has derivative almost everywhere on the interval X and

$$H'(x) = f(x)G(x) + F(x)g(x).$$

Using Theorem 17.2 we get that the function H' is in $L(v, Y)$ and applying Newton's formula we get

$$\int_a^b f(t)G(t) dt + \int_a^b F(t)g(t) dt = F(b)G(b) - F(a)G(a).$$

□

31. TENSOR PRODUCT OF MEASURE SPACES

Assume now that we have two measure spaces (X_i, V_i, v_i) each defined on the prering V_i of an abstract space X_i for $i = 1, 2$. Consider the Cartesian product $X_1 \times X_2$. On the tensor product $V_1 \otimes V_2$ of the families V_i of sets

$$V_1 \otimes V_2 = \{A_1 \times A_2 : A_1 \in V_1 \text{ and } A_2 \in V_2\}$$

define the set function by

$$v_1 \otimes v_2(A_1 \times A_2) = v_1(A_1)v_2(A_2) \quad \text{for all } A_i \in V_i (i = 1, 2).$$

Theorem 31.1 (Tensor product of measure spaces forms measure space). *Assume that (X_i, V_i, v_i) are measure spaces for $i = 1, 2$. Let the triple (X, V, v) consist of $X = X_1 \times X_2$, $V = V_1 \otimes V_2$, and $v_1 \otimes v_2(A_1 \times A_2) = v_1(A_1)v_2(A_2)$ for all $A = A_1 \times A_2 \in V$.*

Then the triple

$$(X, V, v) = (X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2)$$

forms a positive measure space.

Proof. As we established before the tensor product of prerings forms a prering. To prove that the set function $v = v_1 \otimes v_2$ is σ -additive take any set $A \times B$ in V and let $A_n \times B_n \in V$ denote a sequence of disjoint sets whose union is the set $A \times B$. Notice the identity

$$(31.1) \quad c_A(x_1) c_B(x_2) = \sum_n c_{A_n}(x_1) c_{B_n}(x_2) \quad \text{for all } x_1 \in X_1, x_2 \in X_2.$$

Fixing x_2 and integrating with respect to v_1 both sides of the equation (31.1) on the basis of Beppo Levi's theorem for a series 16.8, we get

$$v_1(A) c_B(x_2) = \sum_n v_1(A_n) c_{B_n}(x_2) \quad \text{for all } x_2 \in X_2.$$

Integrating the above term by term with respect to v_2 and applying again Beppo Levi's theorem for a series 16.8 we get

$$v_1(A) v_2(B) = \sum_n v_1(A_n) v_2(B_n)$$

that is the set function

$$v(A \times B) = v_1 \otimes v_2(A \times B) = v_1(A)v_2(B) \quad \text{for all } A \times B \in V_1 \otimes V_2$$

is σ -additive. Hence the triple

$$(X, V, v) = (X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2)$$

forms a measure space. \square

The above theorem has an immediate generalization to any finite number of measure spaces.

Theorem 31.2 (Tensor product of n measure spaces). *Assume that (X_i, V_i, v_i) are measures over abstract spaces X_i and V_i are prerings for $i = 1, \dots, n$. Let the triple (X, V, v) consist of $X = X_1 \times \dots \times X_n$, $V = V_1 \otimes \dots \otimes V_n$, and*

$$v(A) = v_1 \otimes \dots \otimes v_n(A) = v_1(A_1) \cdots v_n(A_n)$$

for all $A = A_1 \times \dots \times A_n \in V$. Then the triple (X, V, v) forms a measure space.

32. FUBINI'S THEOREMS FOR LEBESGUE AND BOCHNER INTEGRALS

Let Y be a Banach space with norm $|\cdot|$. Assume that (X_i, V_i, v_i) for $i = 1, 2$ are measure spaces each on its prerings V_i of the abstract space X_i . For the sake of brevity let (X, V, v) denote the product measure space

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

We shall adopt the following convention for a function f of the point (x_1, x_2) from the product space $X_1 \times X_2$. When x_1 is fixed, by the symbol $f(x_1, \cdot)$ we shall understand the function $x_2 \mapsto f(x_1, x_2)$. To indicate the variable of integration in expressions depending on the variable x we shall write

$$\int f dv = \int f(x) v(dx).$$

We shall also use abbreviation v_1 -a.e. to indicate that the relation preceding it holds almost everywhere with respect to measure v_1 .

Definition 32.1 (Space $Fub(Y)$). *Denote by $Fub(Y)$ the set of Bochner summable functions $f \in L(v, Y)$, for which the **Fubini theorem** is true, that is such that there exists a v_1 -null set C and a summable function $h \in L(v_1, Y)$ such that*

$$(32.1) \quad f(x_1, \cdot) \in L(v_2, Y) \quad \text{and} \quad h(x_1) = \int f(x_1, x_2) v_2(dx_2) \quad \text{if} \quad x_1 \notin C,$$

and moreover

$$(32.2) \quad \int f dv = \int h dv_1 = \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1).$$

We shall prove that

$$Fub(Y) = L(v_1 \otimes v_2, Y)$$

using an argument similar to the argument of S. Saks [25] who used it to prove the theorem for Lebesgue integrals generated by Lebesgue measures over sigma rings. For a reference to the original theorem see Fubini [15].

From linearity of the integrals follows that the set $Fub(Y)$ is linear. Since for any fixed $y \in Y$ and a function of the form

$$s(x_1, x_2) = y c_{A_1 \times A_2}(x_1, x_2) = y c_{A_1}(x_1) c_{A_2}(x_2) \quad \text{for all } (x_1, x_2) \in X_1 \times X_2$$

we have

$$\int s \, dv = \int y c_{A_1}(x_1) c_{A_2}(x_2) \, dv = y v_1(A_1) v_2(A_2) = \int \left(\int y c_{A_2} \, dv_2 \right) c_{A_1} \, dv_1,$$

every simple function s is in the set $Fub(Y)$, that is we have $S(V, Y) \subset Fub(Y)$.

Lemma 32.2 ($Fub(R)$ is closed under monotone pointwise convergence). *Assume that functions f_n represent a sequence monotone with respect to the relation less or equal everywhere on the Cartesian product $X = X_1 \times X_2$.*

If the functions f_n belong to the set $Fub(R)$, and the sequence f_n converges everywhere to a finite-valued function f , and the sequence of integrals $\int f_n \, dv$ is bounded, then the limit function f also belongs to the set $Fub(R)$.

Proof. From the assumption of the lemma there exist v_1 -null sets C_m and functions $h_m \in L(v_1, R)$ such that $f_m(x_1, \cdot) \in L(v_2, R)$ and $h_m(x_1) = \int f_m(x_1, \cdot) \, dv_2$ if $x_1 \notin C_m$ and

$$\int h_m \, dv_1 = \int f_m \, dv \quad \text{for } m = 1, 2, \dots$$

Put $C = \bigcup_m C_m$. Since C is a v_1 -null set, the sequence h_m is monotone with respect to the relation less or equal v_1 -almost everywhere.

Therefore from the Monotone Convergence Theorem 16.7 we get that there exist a function $h \in L(v_1, R)$ and a v_1 -null set D such that

$$h_m(x_1) \rightarrow h(x_1) \quad \text{if } x_1 \notin D$$

and h_m converges to h in the topology of the space $L(v_1, R)$. This implies

$$\int h_m \, dv_1 \rightarrow \int h \, dv_1.$$

For any x_1 which is not in the set $C \cup D$, the sequence of functions $f_m(x_1, \cdot)$, converges monotonically at each point to the function $f(x_1, \cdot)$. Therefore again, for a fixed $x_1 \notin C \cup D$, from the Monotone Convergence Theorem 16.7, and the fact that the sequence of integrals

$$\int f_m(x_1, \cdot) \, dv_2 = h_m(x_1)$$

is bounded, since it is convergent, we get

$$f(x_1, \cdot) \in L(v_2, R)$$

and the equality

$$h(x_1) = \lim_m h_m(x_1) = \lim_m \int f_m(x_1, \cdot) \, dv_2 = \int f(x_1, \cdot) \, dv_2 \quad \text{if } x_1 \notin C \cup D.$$

Since by assumption the sequence f_m converges monotonically everywhere to the function f and the sequence of integrals $\int f_m \, dv$ is bounded, therefore we have

$$f \in L(v, R) \quad \text{and} \quad \int f_m \, dv \rightarrow \int f \, dv.$$

Thus we get

$$\int h \, dv_1 = \lim_m \int h_m \, dv_1 = \lim_m \int f_m \, dv = \int f \, dv.$$

Hence $f \in Fub(R)$. \square

Definition 32.3 (Section of a set $A \subset X_1 \times X_2$). *By a section of a set $A \subset X_1 \times X_2$ corresponding to a fixed value of the variable $x_1 \in X_1$ we shall understand the set*

$$A(x_1) = \{x_2 \in X_2 : (x_1, x_2) \in A\}.$$

Similarly we can define a section of the set for any fixed $x_2 \in X_2$.

Lemma 32.4 (Almost all sections of a v -null set are v_2 -null sets). *Assume that (X_i, V_i, v_i) for $i = 1, 2$ are measure spaces each over its prering V_i of an abstract space X_i . Let (X, V, v) denote the product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

If $A \subset X$ is a v -null set, then there exists a v_1 -null set $C \subset X_1$ such that for every $x_1 \notin C$ the section

$$A(x_1) = \{x_2 \in X_2 : (x_1, x_2) \in A\}$$

of the set A represents a v_2 -null set.

Proof. Let A be a v -null set. One can prove from the definition of a null set, that there exists a sequence of sets $A_n \in V$ such that

$$A \subset \bigcup_{n>m} A_n \quad \text{for } m = 1, 2, \dots$$

and

$$\sum_{n=1}^{\infty} v(A_n) \leq 1.$$

Put

$$B_{mn} = \bigcup_{m < j < m+n} A_j, \quad B_m = \bigcup_{n>m} A_n \quad \text{and} \quad B = \bigcap_m B_m.$$

We see that $A \subset B$. Consider the characteristic function c_{B_m} . Notice that since $V = V_1 \otimes V_2$ is a prering the family $S(V)$ of simple sets forms a ring. Since the set B_{mn} is a finite union of sets from the prering V we must have $B_{mn} \in S(V)$. Hence the characteristic functions $c_{B_{mn}}$ are in the space $S(V, R)$ of simple functions. Now when $n \rightarrow \infty$ the sequence converges increasingly everywhere to the function c_{B_m} . Since for fixed m the integrals of the functions $c_{B_{mn}}$ are bounded by the number

$$\sum_{n>m} v(A_n) \quad \text{for } m = 1, 2, \dots$$

representing the remainder of a convergent series, we get $c_{B_m} \in Fub(R)$ for all m and

$$\int c_{B_m} \, dv \leq \sum_{n>m} v(A_n) \quad \text{for } m = 1, 2, \dots$$

When $m \rightarrow \infty$ the sequence c_{B_m} converges monotonically everywhere to the function c_B . Thus from Lemma 32.2 we get $c_B \in Fub(R)$, that is there exists a function $h \in L(v_1, R)$ and a v_1 -null set C_1 such that $c_B(x_1, \cdot) \in L(v_2, R)$

$$h(x_1) = \int c_B(x_1, \cdot) dv_2 \quad \text{if} \quad x_1 \notin C_1.$$

Moreover we have

$$\int h dv_1 = \int c_B dv \leq \int c_{B_m} dv \leq \sum_{n > m} v(A_n) \quad \text{for all} \quad m.$$

Since the function c_B is non-negative, therefore we have $h(x_1) \geq 0$ v_1 -almost everywhere. This yields

$$\|h\| = \int |h| dv_1 = \int h dv_1.$$

Now from Theorem 16.2, concerning the basic properties of the space $L(v, Y)$, we get that there exists a v_1 -null set C_2 such that $h(x_1) = 0$ if $x_1 \notin C_2$.

Put $C = C_1 \cup C_2$. We see that

$$0 = \int c_B(x_1, \cdot) dv_2 = \|c_B(x_1, \cdot)\|_{v_2} \quad \text{if} \quad x_1 \notin C.$$

This yields

$$c_B(x_1, x_2) = 0 \quad \text{if} \quad x_2 \notin C(x_1),$$

where $C(x_1)$ is a v_2 -null set. Since we have the equality

$$c_B(x_1, x_2) = c_{B(x_1)}(x_2) \quad \text{for all} \quad x_1 \in X_1, x_2 \in X_2,$$

where

$$B(x_1) = \{x_2 \in X_2 : (x_1, x_2) \in B\},$$

we must have

$$B(x_1) \subset C(x_1).$$

Thus the set $B(x_1)$ is a v_2 -null set and so is the set

$$A(x_1) \subset B(x_1).$$

Thus the Lemma is proved. \square

Using Lemma 32.4 by a similar argument as in Lemma 32.2 we can prove the following:

Lemma 32.5 ($Fub(R)$ is closed under monotone convergence v -a.e.). *Assume that functions f_n represent a sequence monotone with respect to the relation less or equal v -almost everywhere on the product space $X = X_1 \times X_2$.*

If the functions f_n belong to the set $Fub(R)$, and the sequence f_n converges v -almost everywhere to a finite-valued function f , and the sequence of integrals $\int f_n dv$ is bounded, then the limit function f also belongs to the set $Fub(R)$.

Theorem 32.6 (Fubini-Bochner: $Fub(Y) = L(v, Y)$). *Assume that Y is a Banach space and (X_i, V_i, v_i) for $i = 1, 2$ are measure spaces over prerings V_i . Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

Then every Bochner summable function $f \in L(v, Y)$ belongs to the family $Fub(Y)$, that is there exists a v_1 -null set C and a summable function $h \in L(v_1, Y)$ such that

$$f(x_1, \cdot) \in L(v_2, Y) \quad \text{and} \quad h(x_1) = \int f(x_1, x_2) v_2(dx_2) \quad \text{if} \quad x_1 \notin C,$$

and moreover

$$\int f dv = \int h dv_1 = \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1).$$

Proof. Take any Bochner summable function $f \in L(v, Y)$. It follows from the definition of the space of summable functions that there exists a basic sequence s_n convergent v -almost everywhere to the function f . By the definition of a basic sequence we have

$$s_n = h_1 + \cdots + h_n, \quad \|h_n\|_v \leq M 4^{-n} \quad \text{for} \quad n = 1, 2, \dots,$$

where $h_n \in S(V, Y)$ represents a sequence of simple functions.

Consider the sequence of real-valued functions

$$g_n(x) = |h_1(x)| + \cdots + |h_n(x)| \quad \text{for} \quad x \in X.$$

We see that

$$g_n \in S(V, R) \quad \text{and} \quad \int g_n dv \leq M \quad \text{for} \quad n = 1, 2, \dots$$

Since the sequence g_n is monotone and the sequence of integrals is bounded, there exists a Lebesgue summable function $g \in L(v, R)$ such that the sequence g_n converges v -almost everywhere to the function g . Since $g_n \in Fub(R)$ therefore $g \in Fub(R)$ in accord with Lemma 32.5.

Notice that

$$|s_n(x)| \leq g(x) \quad \text{for} \quad n = 1, 2, \dots$$

v -almost everywhere. Therefore there exists a v -null set A such that

$$|s_n(x)| \leq g(x) \quad \text{for} \quad x \notin A \quad \text{and} \quad n = 1, 2, \dots$$

and

$$s_n(x) \rightarrow f(x) \quad \text{if} \quad x \notin A.$$

Let C_1 be a v_1 -null set such that the sections $A(x_1)$ of the set A are v_2 -null sets if $x_1 \notin C_1$.

Since $g \in Fub(R)$ there exists a v_1 -null set C_2 and a function $h \in L(v_1, R)$ such that

$$g(x_1, \cdot) \in L(v_2, R) \quad \text{and} \quad h(x_1) = \int g(x_1, \cdot) dv_2 \quad \text{if} \quad x_1 \notin C_2$$

and

$$\int h dv_1 = \int g dv.$$

Take any point

$$x_1 \notin C_1 \cup C_2 = C.$$

We have

$$|s_n(x_1, x_2)| \leq g(x_1, x_2) \quad \text{for } x_2 \notin A(x_1) \quad \text{and } n = 1, 2, \dots$$

Thus the sequence $s_n(x_1, \cdot)$ is dominated by a summable function and converges v_2 -almost everywhere to the function $f(x_1, \cdot)$. Therefore from the Dominated Convergence Theorem 16.11 we get

$$f(x_1, \cdot) \in L(v_2, R)$$

and

$$\tilde{s}_n(x_1) = \int s_n(x_1, \cdot) dv_2 \rightarrow \int f(x_1, \cdot) dv_2.$$

Notice the estimate

$$|\tilde{s}_n(x_1)| \leq \int |s_n(x_1, \cdot)| dv_2 \leq \int g(x_1, \cdot) dv_2 = h(x_1) \quad \text{if } x_1 \notin C.$$

The function defined by the formula

$$\tilde{f}(x_1) = \int f(x_1, \cdot) dv_2 \quad \text{if } x_1 \notin C, \quad \tilde{f}(x_1) = 0 \quad \text{if } x_1 \in C,$$

being the limit almost everywhere of the sequence of simple functions \tilde{s}_n , dominated by the summable function h , is summable, that is $\tilde{f} \in L(v_1, Y)$. Moreover

$$\int \tilde{f} dv_1 = \lim_n \int \tilde{s}_n dv_1 = \lim_n \int s_n dv = \int f dv.$$

Thus we have proved that $L(v, Y) = Fub(Y)$. \square

It is convenient to formulate the Fubini theorem in a more concise way. Introduce first the following notion.

Definition 32.7 (Meaningful iterated integral). *Assume that we are given two positive measure spaces (X_i, V_i, v_i) on prerings V_i of some abstract spaces X_i and a Banach space Y . Let (X, V, v) denote the tensor product of these measure spaces.*

*We shall say that the iterated integral for a function $f : X \mapsto Y$ is **meaningful***

$$\int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1)$$

if the inner integral exists as Bochner integral and its value $h(x_1)$ for v_1 -almost all points x_1 yields a Bochner summable function $h \in L(v_1, Y)$.

Theorem 32.8 (Fubini theorem). *Assume that we are given two positive measure spaces (X_i, V_i, v_i) on prerings V_i of some abstract spaces X_i and a Banach space Y . Let (X, V, v) denote the tensor product of these measure spaces.*

If f belongs to the space $L(v, Y)$ of Bochner summable functions then the following iterated integrals are meaningful and we have the equalities

$$\int f dv = \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1) = \int \left(\int f(x_1, x_2) v_1(dx_1) \right) v_2(dx_2)$$

33. FUBINI THEOREM IN TERMS OF EQUIVALENCE CLASSES

As before assume that X is an abstract space and Y a Banach space. We shall use the notation $\|\cdot\|$ to denote the norm in Y . Assume that (X, V, ν) is a measure space over a premeasurable V of sets of X . The functional $f \mapsto \|f\|$ considered on the space $L(\nu, Y)$ represents a semi-norm. By identifying functions equal almost everywhere we may obtain a Banach space.

To perform this identification formally proceed as follows. Let $F(X, Y)$ denote the space of all functions from the space X into the space Y .

Definition 33.1 (Equivalence relation generated by equality almost everywhere). Define the relation \equiv between functions in $F(X, Y)$ by the condition

$$(f \equiv g) \iff (f(x) = g(x) \quad \nu - \text{almost everywhere}).$$

This relation is an **equivalence relation** and thus it splits the space $F(X, Y)$ into disjoint classes of functions that are equal almost everywhere. Denote by $[f]$ the class of functions containing the function f . Now if $L \subset F(X, Y)$ is any sub-collection of functions, let L_0 denote the collection of classes generated by this equivalence relation, that is

$$L_0 = \{[f] : f \in L\}.$$

In particular $L_0(\nu, Y)$ will denote the space of classes that are generated by the space $L(\nu, Y)$ of Bochner summable functions, and $M_0(\nu, Y)$ the collection generated by the space $M(\nu, Y)$ of Bochner measurable functions as defined in Bogdanowicz [5].

For $p > 0$ let $L_0^p(\nu, Y)$ denote the collection generated by the space $L^p(\nu, Y)$, where

$$L^p(\nu, Y) = \left\{ f \in M(\nu, Y) : \int |f(x)|^p \nu(dx) < \infty \right\}.$$

It follows from Theorem 16.2 that the functional

$$\|g\|_v = \int |f| \, d\nu \quad \text{if } f \in g \in L_0(\nu, Y).$$

is well defined and represents a norm on the space $L_0(\nu, Y)$. Moreover the pair

$$(L_0(\nu, Y), \|\cdot\|_v)$$

forms a Banach space.

Define the integral operator on the space $L_0(\nu, Y)$ by

$$\int g \, d\nu = \int f \, d\nu \quad \text{if } g \in L_0(\nu, Y) \quad \text{and} \quad [f] = g.$$

From the linearity of the integral operator $\int f \, d\nu$ and from the estimate

$$\left| \int f \, d\nu \right| \leq \|f\|_v \quad \text{for all } f \in L(\nu, Y)$$

we get that the integral operator on the space $L_0(\nu, Y)$ is well defined. Indeed, if $[f_1] = [f_2]$ that is $f_1(x) = f_2(x) \quad \nu - a.e.$ we have

$$\int f_1 \, d\nu = \int f_2 \, d\nu$$

according to Theorem 16.2. The operator $\int g dv$ is linear on the space $L_0(v, Y)$ and

$$\left| \int g dv \right| \leq \|g\|_v \quad \text{for all } g \in L_0(v, Y).$$

It will be convenient to use also the following notation when it is important to indicate the variable of integration

$$\int g(x) v(dx) = \int g dv.$$

Now consider the space $L_0(v, Y)$ for the product measure $v = v_1 \otimes v_2$. Take a function

$$f \in L_0(v, Y).$$

We shall say that the double integral

$$I = \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1)$$

has a meaning if for every function $g \in L(v, Y)$ being a representative of the class f there exists a v_1 -null set C and a function

$$h \in L(v_1, Y)$$

such that the function $g(x_1, x_2)$ considered as the function of the variable x_2 is v_2 -summable if $x_1 \notin C$ and

$$h(x_1) = \int g(x_1, x_2) v_2(dx_2) \quad (x_1 \notin C),$$

and by the value of the double integral we shall understand the value

$$I = \int h dv_1.$$

It follows from the next theorem that this definition is correct. It will be convenient to use the following notation

$$\int f dv = \int f(x_1, x_2) v_1(dx_1) v_2(dx_2)$$

for an integral generated by the product measure.

Theorem 33.2. *Assume that Y is a Banach space and X_i are abstract spaces with prerings V_i . Assume that (X_i, V_i, v_i) for $i = 1, 2$ form positive measure spaces. Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

Then for every function

$$f \in L_0(v, Y)$$

the iterated integrals in the following formula have a meaning and they are equal

$$\begin{aligned} \int f(x_1, x_2) v_1(dx_1) v_2(dx_2) &= \int \left(\int f(x_1, x_2) v_1(dx_1) \right) v_2(dx_2) \\ &= \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1). \end{aligned}$$

Proof. The proof of the theorem follows immediately from the above definitions and from Fubini-Bochner Theorem 32.6. \square

Now let us find the relations between the completions $v_c, (v_1)_c, (v_2)_c$ of the measures $v = v_1 \otimes v_2, v_1, v_2$. We remind the reader that if v is a measure on a prepring V of an abstract space X then the completion v_c is defined on the family of summable sets

$$V_c = \{A \subset X : c_A \in L(v, R)\}$$

by the formula

$$v_c(A) = \int c_A dv = \|c_A\|_v.$$

Theorem 33.3. *Assume that X_i are abstract spaces with prerings V_i . Assume that (X_i, V_i, v_i) for $i = 1, 2$ form positive measure spaces. Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

Then for every summable set $A \subset V_c$ there exists v_1 -null set C such that

$$A(x_1) \in (V_2)_c \quad \text{if} \quad x_1 \notin C$$

and the function h given by the formula

$$h(x_1) = (v_2)_c(A(x_1)) \quad \text{if} \quad x_1 \notin C$$

belongs to the space $L_0(v_1, R)$ and

$$v_c(A) = \int (v_2)_c(A(x_1)) dv_1.$$

Proof. Consider the function

$$f(x_1, x_2) = c_A(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in X_1 \times X_2.$$

Notice that

$$f(x_1, x_2) = c_{A(x_1)}(x_2) \quad \text{for all} \quad (x_1, x_2) \in X_1 \times X_2.$$

Using Fubini-Bochner Theorem 32.6 and the definition of the completion of a measure on a prepring, we can conclude the proof. \square

If V is a family of sets denote by V^σ the family of all sets of the form

$$A = \bigcup_n A_n$$

where A_n is a sequence of sets from the family V .

Theorem 33.4. *$[\{A \text{ is } v_1\text{-null} \ \& \ B \in (V_2)^\sigma\} \implies A \times B \text{ is } v\text{-null}]$ Assume that X_i are abstract spaces with prerings V_i . Assume that (X_i, V_i, v_i) for $i = 1, 2$ form positive measure spaces. Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

If the set A is a v_1 -null set and $B \in (V_2)^\sigma$ then the set $A \times B$ is a v -null set.

Proof. Take any sequence of sets $B_n \in V_2$ and let $B = \bigcup_n B_n$ denote their union. The set A is a v_1 -null set if and only if there exists a sequence of sets $A_n \in V_1$ such that

$$\sum_{n=1}^{\infty} v_1(A_n) < \infty$$

and

$$A \subset \bigcup_{n>m} A_n \quad \text{for } m = 1, 2, \dots$$

Notice the equality

$$A \times B = \bigcup_n A \times B_n$$

Consider a fixed set $A \times B_n$. We have

$$A \times B_n \subset \bigcup_{k>m} A_k \times B_n \quad \text{for } m = 1, 2, \dots$$

and

$$\sum_{k>m} v(A_k \times B_n) = \sum_{k>m} v_1(A_k) v_2(B_n) = v_2(B_n) \sum_{k>m} v_1(A_k) \rightarrow 0$$

as $m \rightarrow \infty$. This implies that the set $A \times B_n$ is a v -null set. Thus the set $A \times B$ as the union of a countable number of v -null sets is a v -null set. \square

Definition 33.5 (Family $N(v, Y)$ of null functions). *If Y is a Banach space and (X, V, v) is a measure space over a prerings V let*

$$N(v, Y) = \left\{ f \in L(v, Y) : \int |f(x)| v(dx) = 0 \right\}.$$

*Functions belonging to this family will be called **null functions**.*

Corollary 33.6. *Assume that X_i are abstract spaces with prerings V_i . Assume that (X_i, V_i, v_i) for $i = 1, 2$ form positive measure spaces. Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

If $f_1 \in L(v_1, R)$ and $f_2 \in N(v_2, R)$ then the function f defined by the formula

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad \text{for } (x_1, x_2) \in X_1 \times X_2$$

belongs to the set $N(v, R)$.

Proof. It follows from the definition of the space $L(v, R)$ of summable functions that for every function

$$f_1 \in L(v_1, R)$$

there exists a set $A \in (V_1)^\sigma$ such that

$$f_1(x) = 0 \quad \text{if } x \notin A.$$

From the definition of a null function we have

$$f_2(x) = 0 \quad \text{if } x \notin B$$

for some v_2 -null set B . This implies

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) = 0 \quad \text{if } (x_1, x_2) \notin A \times B.$$

According to Theorem 33.4 the set $A \times B$ is v -null set. Thus we have

$$f \in N(v, R).$$

This concludes the proof of the corollary. \square

Let v be a measure on a prering V . Denote by

$$S = S(V)$$

the family of simple sets generated by the prering. Let S_δ be the family of all sets of the form

$$A = \bigcap_n A_n$$

where A_n is a sequence of simple sets. It is easy to see that

$$S_\delta \subset V_c$$

Denote by V_v the family of all sets of the form

$$A = \bigcup_n A_n$$

where A_n is an increasing sequence of sets from the family S_δ such that the sequence of numbers $v_c(A_n)$ is bounded. Let N_v denote the family of all v -null sets.

It follows from a result in the paper of Bogdanowicz [5], page 259, Theorem 4, Part 6, that a set A is summable if and only if there exist a set $B \in V_v$ and a set $C \in N_v$ such that the set A can be represented as the symmetric difference

$$A = B \div C = (B \setminus C) \cup (C \setminus B).$$

Theorem 33.7. *Assume that the triples (X, W, w) and (Y, U, u) represent measure spaces over prerings W, U , respectively. Let (Z, V, v) denote the product measure space*

$$(X \times Y, W \otimes U, w \otimes u).$$

Then the measure v_c is an extension of the product measure $\rho = w_c \otimes u_c$ and moreover $\rho_c = v_c$.

Proof. Denote by V, W, U , respectively, the domains of the measures v, w, u . Take two sets $A \in W_c$ and $B \in U_c$. These sets can be represented in the form

$$A = C \div D \quad \text{and} \quad B = E \div F$$

where

$$C \in W_w, \quad D \in N_w, \quad E \in U_u, \quad F \in N_u.$$

It is plain that

$$G \in V_v \subset V_c, \quad \text{where} \quad G = C \times E.$$

This implies

$$c_G \in L(v, R).$$

Notice that the condition $A = C \div D$ is equivalent to $A \div C = D$ which in turn is equivalent to the condition $c_A - c_C = h \in N(w, R)$. Similarly one can prove that

$$c_B - c_E = g \in N(u, R).$$

Using the identity

$$c_{A \times B}(x_1, x_2) = c_A(x_1)c_B(x_2) \quad \text{for all } x_1 \in X_1, x_2 \in X_2$$

we easily get the representation

$$c_{A \times B} = c_{C \times E} + k$$

where

$$k = hg + hc_E + gc_C.$$

It follows from Corollary 1 that the function k is a v -null function. This means that the functions

$$c_{A \times B}, c_{C \times E} = c_G$$

are equal v -almost everywhere. Since the function c_G is a v -summable function, therefore according to Theorem 16.2 the function $c_{A \times B}$ is also summable, that is we have

$$A \times B \in V_c.$$

Now from Fubini-Bochner Theorem 32.6 we get

$$w_c(A)u_c(B) = \int c_{A \times B} dv = v_c(A \times B) \quad \text{for all } A \times B \in V.$$

The relation $\eta \subset \mu$ between two functions will mean that the graph of the function η is a subset of the graph of the function μ . Thus we have proved

$$\rho = w_c \cdot u_c \subset v_c.$$

Since the measure ρ_c is the smallest complete measure being an extension of the measure ρ and the measure v_c is complete we get from the previous relation the relation $\rho_c \subset v_c$. Similarly from the relation $v \subset \rho \subset \rho_c$ we get $v_c \subset \rho_c$. This proves $\rho = v_c$. \square

Since the completions of the measures v , ρ , v_c coincide therefore according to Bogdanowicz [5], page 267, Section 7, Theorem 8, they generate the same Lebesgue integral as the product measure.

Corollary 33.8. *The measures: $v = w \otimes u$, $\rho = w_c \otimes u_c$, and v_c , all generate the same Lebesgue-Bochner integration.*

34. TONELLI'S THEOREM FOR THE EXTENDED LEBESGUE INTEGRAL.

Let X be an abstract space with a prering V of sets and Y a Banach space. Assume that the triple (X, V, v) forms a positive measure space.

We define the space $M(v, Y)$ of **Bochner measurable functions** as in Bogdanowicz [5]. A function $f : X \rightarrow Y$ belongs to the space $M(v, Y)$ if it has its support in a set $B \in V^\sigma$, that is

$$f(x) = 0 \quad \text{if} \quad x \notin B,$$

and

$$c_A a(f(\cdot)) \in L(v, Y) \quad \text{for all} \quad A \in V,$$

where

$$a(y) = (1 + |y|)^{-1} y \quad \text{for all} \quad y \in Y.$$

This definition allows us, using the properties of the space of Lebesgue-Bochner summable functions $L(v, Y)$ to get the classical properties of, and relations between, the spaces

$$L(v, Y), \quad M(v, Y), \quad L(v, R), \quad M(v, R).$$

For details see Bogdanowicz [5].

Let $M^+(v)$ denote the space of all functions f from the set X into the closed extended interval $[0, \infty]$ for which there exist a set $B \in V_\sigma$ such that

$$f(x) = 0 \quad \text{if} \quad x \notin B,$$

and

$$c_A b(f(\cdot)) \in L(v, R) \quad \text{for all} \quad A \in V,$$

where

$$b(y) = (1 + y)^{-1} y \quad \text{if} \quad y \in [0, \infty), \quad \text{and} \quad b(\infty) = 1.$$

Functions belonging to the space $M^+(v)$ will be called Lebesgue measurable extended functions.

It was proven in [5] that $f \in M^+(v)$ if and only if there exist non-negative Lebesgue summable functions $f_n \in L(v, R)$ such that the sequence f_n increasingly converges almost everywhere to the function f . This allows us to extend the integral onto $M^+(v)$ by the formula

$$\int f \, dv = \lim_n \int f_n \, dv.$$

It follows from the Monotone Convergence Theorem that this extension of the integral onto the space $M^+(v)$ does not depend on the choice of the sequence f_n . Thus the definition is correct.

The following implications have been proved in [5]

$$f \in M(v, Y) \implies |f(\cdot)| \in M^+(v)$$

and

$$\{f \in M(v, Y) \text{ and } \int |f(\cdot)| \, dv < \infty\} \implies f \in L(v, Y).$$

Theorem 34.1 (Tonelli). *Assume that X_i are abstract spaces with prerings V_i . Assume that (X_i, V_i, v_i) for $i = 1, 2$ form positive measure spaces. Let (X, V, v) denote the tensor product measure space*

$$(X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2).$$

Then for every extended measurable function $f \in M^+(v)$ there exists a v_1 -null set C and an extended measurable function $h \in M^+(v_1)$ such that

$$f(x_1, \cdot) \in M^+(v_2) \quad \text{and} \quad h(x_1) = \int f(x_1, x_2) v_2(dx_2) \quad \text{if} \quad x_1 \notin C,$$

and moreover

$$\int f dv = \int h dv_1 = \int \left(\int f(x_1, x_2) v_2(dx_2) \right) v_1(dx_1).$$

Proof. Take any function $f \in M^+(v)$. There exists a sequence of non-negative Lebesgue summable functions $f_n \in L(v, R)$ that increasingly converges to the function f except on a v -null set A . Denote by C_0 a v_1 -null set such that for each $x_1 \notin C_0$, the section $A(x_1)$ of the set A forms a v_2 -null set.

It follows from Fubini's theorem that there exist functions $h_n \in L(v_1, R)$ and a sequence C_n of v_1 -null sets such that

$$f_n(x_1, \cdot) \in L(v_2, R) \quad \text{and} \quad h_n(x_1) = \int f_n(x_1, x_2) dv_2 \quad \text{if} \quad x_1 \notin C_n.$$

It is easy to see that one may assume that the functions h_n are non-negative and

$$\int h_n dv_1 = \int f_n dv \quad \text{for all} \quad n = 1, 2, \dots$$

Put

$$C = \bigcup_{n=0}^{\infty} C_n.$$

If $x_1 \notin C$ then the sequence of functions $f_n(x_1, \cdot)$ increasingly converges to the function $f(x_1, \cdot)$ for $x_2 \notin A(x_1)$, where $A(x_1)$ represents a v_2 -null set. This implies

$$f(x_1, \cdot) \in M^+(v_2)$$

and

$$\int f(x_1, \cdot) dv_2 = \lim_n \int f_n(x_1, \cdot) dv_2.$$

If $x_1 \notin C$ then the sequence

$$h_n(x_1) = \int f_n(x_1, \cdot) dv_2$$

increasingly converges to a value $h(x_1) \in [0, \infty]$. Put $h(x_1) = 0$ for $x_1 \in C$. Notice that

$$h(x) = \lim_n h_n(x_1) = \lim_n \int f_n(x_1, \cdot) dv_2 = \int f(x_1, \cdot) dv_2 \quad \text{if} \quad x_1 \notin C.$$

Since the sequence $h_n \in L(v_1, R)$ increasingly converges almost everywhere to the function h therefore we have $h \in M^+(v_1)$ and

$$\int h dv_1 = \lim_n \int h_n dv_1 = \lim_n \int f_n dv = \int f dv.$$

Thus the theorem is proved. \square

35. EXTENSION OF VECTOR MEASURE

Now let us consider the case when the space Y is the space R of reals and Z any Banach space and the bilinear operator u is the multiplication operator $u(r, z) = rz$.

Proposition 35.1 (Isomorphism and isometry of $K(v, Z)$ and $K(v_c, Z)$). *Assume that (X, V, v) is a measure space on a pre-ring V of subsets of an abstract space X .*

Every vector measure $\mu \in K(v, Z)$ can be extended from the pre-ring V onto the delta ring V_c of summable sets by the formula

$$\mu_c(A) = \int u(c_A, d\mu) \quad \text{for all } A \in V_c.$$

This extension establishes isometry and isomorphism between the Banach spaces $K(v, Z)$ and $K(v_c, Z)$.

36. ABSOLUTE CONTINUITY OF VECTOR MEASURES

Definition 36.1 (Absolute continuity of a vector measure). *Given a Lebesgue measure space (X, V, v) on a σ -algebra V and σ -additive vector measure μ from V into a Banach space Y . We say that the vector measure μ is absolutely continuous with respect to the measure v if*

$$\mu(A) = 0 \quad \text{whenever } v(A) = 0.$$

Theorem 36.2 (Phillips). *Assume that (X, V, v) is a Lebesgue measure space on a σ -algebra V such that $v(X) < \infty$ and Y is a reflexive Banach space.*

Assume that μ is a σ -additive vector measure of finite variation from the σ -algebra V into Y .

If μ is absolutely continuous with respect to the Lebesgue measure v , then there exists a Bochner summable function $g \in L(v, Y)$ such that

$$\mu(A) = \int_A g dv \quad \text{for all } A \in V.$$

For the proof of this remarkable theorem see Diestel and Uhl [11, page 76]. This result can be found in the original paper of Phillips [23].

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